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THE

MESSENGER OF MATHEMATICS,

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VOL. XI.

[MAY, 1881—APRIL, 1882.]

MACMILLAN AND CO.

London and Cambridge.

EDINBURGH: EDMONSTON & CO. GLASGOW: JAMES MACLEHOSE.
DUBLIN: HODGES, FOSTER & CO. OXFORD: JOHN HENRY AND J. PARKER.

1882.

CAMBRIDGE:

PRINTED BY W. METCALFE AND SON, TRINITY STREET.

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NOTE. Pp. 1—16 were published in May, 1881; pp. 17—32 in June 1881; and so on, pp. 177—192 being published in April, 1882; so that the month of publication of any paper may be readily ascertained.

MESSENGER OF MATHEMATICS.

ON SOME DISCONTINUOUS SERIES CONSIDERED BY RIEMANN.

By *H. J. S. Smith*, Savilian Professor of Geometry in the University of Oxford.

RIEMANN, in a fragment published after his death in his collected works,* has determined the value of certain q series in the limiting case in which the analytical modulus of q is unity. If $q = \rho\varepsilon^{i\theta}$, the series considered by Riemann contain θ only, and converge and diverge for an infinite number of values of θ between any two limits however near to one another; they appear, indeed, to have attracted Riemann's attention by this peculiarity. Two of these series, which may be regarded as examples of the rest, are considered in the present note, and some details relating to them are supplied which are omitted in Riemann's brief record of his results.

Let
$$S(q) = 4\sum \frac{(-1)^n}{n} \frac{q^n}{1+q^n} \ldots\ldots\ldots\ldots\ldots\text{(i)},$$

n denoting any positive integral number from $+1$ to ∞, so that (*Fundamenta Nova*, p. 103).

$$S(q) = \log \frac{k}{4\sqrt{q}}.$$

Put $q = \rho\varepsilon^{i\theta}$, multiply each side of (i) by $\frac{dq}{q} = \frac{d\rho}{\rho}$, and take the rectilineal integral from 0 to $\rho\varepsilon^{i\theta}$ along the vector $\varepsilon^{i\theta}$; we have

$$\int_0^{\rho\varepsilon^{i\theta}} S(q)\frac{dq}{q} = 4\sum \frac{(-1)^n}{n^2} \log\left(1+\rho^n\varepsilon^{ni\theta}\right) \ldots\ldots\text{(ii)},$$

* *Mathematische Werke*, p. 427.

the logarithms on the right being so taken as to vanish with ρ; or, which is the same thing,

$$\int_0^{\rho\varepsilon^{i\theta}} S(q)\frac{dq}{q} = 4\sum \frac{(-1)^n \log \Delta_n}{n^2}$$

$$+ 4i\sum \frac{(-1)^n}{n^2} \tan^{-1}\left(\frac{\rho^n \sin n\theta}{1+\rho^n \cos n\theta}\right) \ldots\ldots\ldots\ldots \text{(iii)},$$

where Δ_n is the arithmetical square root of

$$1 + 2\rho^n \cos n\theta + \rho^{2n},$$

$\log \Delta_n$ is a real logarithm, and $\tan^{-1}\left(\frac{\rho^n \sin n\theta}{1+\rho^n \cos n\theta}\right)$ is an angle included between the limits $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. So long as $\rho < 1$, the series $S(q)$, and the integrated series (ii) or (iii) are absolutely convergent. When $\rho = 1$, the series (i) and (iii) respectively become

$$4\sum \frac{(-1)^n}{n} \frac{\varepsilon^{ni\theta}}{1+\varepsilon^{ni\theta}} = -2\log 2 - 2\sum \frac{(-1)^n}{n}$$

$$+ 4\sum \frac{(-1)^n}{n} \frac{\varepsilon^{\frac{1}{2}ni\theta}}{\varepsilon^{\frac{1}{2}ni\theta} + \varepsilon^{-\frac{1}{2}ni\theta}} = -2\log 2 + 2i\sum \frac{(-1)^n}{n} \tan \tfrac{1}{2}(n\theta)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(iv)},$$

and

$$2\sum \frac{(-1)^n}{n^2} \log\{4\cos^2 \tfrac{1}{2}(n\theta)\} + 4i\sum \frac{(-1)^n}{n^2} [\tfrac{1}{2}(n\theta)] \ldots \text{(v)};$$

where by $[\frac{1}{2}(n\theta)]$ we understand the absolutely least angle which has the same tangent as $[\frac{1}{2}(n\theta)]$, except when $[\frac{1}{2}(n\theta)]$ is an uneven multiple of $\frac{1}{2}\pi$, in which case $[\frac{1}{2}(n\theta)] = 0$. The imaginary part of (v) is always convergent; the convergence or divergence of (iv) and of the real part of (v) depends on the nature of the value assigned to θ.

First, let $\frac{\theta}{\pi}$ be incommensurable. In this case the series (iv) is always divergent. For if $\frac{a'}{b'}$, $\frac{a}{b}$ are two consecutive convergents to $\frac{\theta}{\pi}$ of which the latter has an

uneven numerator a, and if ϕ is the complete quotient immediately succeeding $\frac{a}{b}$, we have

$$\tfrac{1}{2}b\theta = \tfrac{1}{2}a\pi \pm \frac{\frac{1}{2}\pi}{b\phi + b'};$$

whence, when b is very great, the absolute value of $\frac{1}{b}\tan\tfrac{1}{2}b\theta$ is approximately $\dfrac{\phi + \frac{b'}{b}}{\frac{1}{2}\pi}$. Thus the series (iv) contains an infinite number of terms of which the absolute value is finite, and it is consequently divergent. Riemann says that when $\frac{\theta}{\pi}$ is incommensurable, the real part of the series (v) is also always divergent. But this appears to be an oversight; for the real part of (v) is in fact divergent or convergent according as the incommensurable quantity θ does, or does not, satisfy certain conditions; and, in particular, it can be shown that, whenever θ is the incommensurable root of an equation of a finite order having integral coefficients, the real part of (v) is convergent. For each successive value of n we form the equation

$$\tfrac{1}{2}n \times \frac{\theta}{\pi} - m = \tfrac{1}{2}\varepsilon(1 - \nu),$$

where m is an integral number, $\varepsilon = \pm 1$, and $0 < \nu < 1$; it is evident that m, ε, and ν can always be determined in one way and in one way only. We then have

$$\log\{4\cos^2\tfrac{1}{2}(n\theta)\} = \log\{4\sin^2\tfrac{1}{2}(\nu\pi)\},$$

and, since

$$\frac{2}{\pi} < \frac{2\sin\frac{1}{2}(\nu\pi)}{\nu\pi} < 1,$$

the series

$$\sum \frac{(-1)^n}{n^2}\log\frac{4\sin^2\frac{1}{2}(\nu\pi)}{\nu^2\pi^2}$$

is absolutely convergent; *i.e.* the series

$$\sum \frac{(-1)^n}{n^2}\log\{4\cos^2\tfrac{1}{2}(n\theta)\} \quad \text{.....(vi)}$$

is convergent or divergent according as the series

$$\sum (-1)^n \frac{\log\nu}{n^2} \quad \text{.....(vii)}$$

is convergent or divergent. Let $\frac{\theta}{\pi}$ be the root of an irreducible equation

$$f(x) = Ax^s + Bx^{s-1} + \ldots Mx + N = 0;$$

of which the coefficients A, B, ... are integral; it may be shown as follows by a method due to M. Liouville, that the series (vii) is convergent.

Let x_1, x_2, ... be the real roots other than $\frac{\theta}{\pi}$ of $f(x)$, $y_1 \pm iz_1$, $y_2 \pm iz_2$, ... its imaginary roots; and let $\delta - 1$ be a positive quantity greater than the greatest of the differences $\frac{\theta}{\pi} - x$, $\frac{\theta}{\pi} - y$; we have

$$n^s f\left(\frac{2m+\varepsilon}{n}\right) = A \times \frac{\varepsilon\nu}{n} \times n^s \times \prod\left(\frac{2m+\varepsilon}{n} - x\right) \times \prod\left\{\left(\frac{2m+\varepsilon}{n} - y\right)^2 + z^2\right\}.$$

But $n^s f\left(\frac{2m+\varepsilon}{n}\right)$ is an integral number, which cannot be zero, because $f(x)$ is irreducible; hence

$$\nu > \frac{1}{A\,\delta^{s-1}\,n^{s-1}},$$

or $-\log\nu < (s-1)\log n + c$, where c is a finite constant; this shows that (vii) is convergent, because $\sum \frac{\log n}{n^2}$ is convergent.

The same reasoning would suffice to prove, that when $\frac{\theta}{\pi}$ is a root of any equation such as $f(x)$, the series

$$\sum \frac{\log\{\cos^2\frac{1}{2}(n\theta)\}}{\phi(n)}$$

always converges if $\sum \frac{\log n}{\phi(n)}$ is absolutely convergent.

On the other hand, between any two limits however near, there is always an infinite number of values of θ, for which the series (vi), and indeed any series of the form

$$\sum \frac{\log\{\cos^2\frac{1}{2}(n\theta)\}}{\phi(n)} \quad \ldots\ldots\ldots\ldots \text{(viii)}$$

becomes divergent, however rapidly the positive function $\phi(n)$ may increase with n. If, for example, in the developement of $\frac{\theta}{\pi}$ in a continued fraction, it happens an infinite number of times that a convergent $\frac{a}{b}$, having an uneven numerator a, is immediately followed by a complete quotient which surpasses $\frac{c}{b}\varepsilon^{\phi(b)}$, where c is any finite positive constant, the series (viii) contains an infinite number of terms of which the absolute value value is finite, and is therefore divergent.

Secondly, let $\frac{\theta}{\pi}=\frac{a}{b}$, a rational fraction in its lowest terms. If a is uneven, the series (iv) and (vi) contain terms which are infinite. But when a is even the two series are convergent, and Riemann has succeeded in assigning their sums in the following manner.

Writing $2a$ for a, and observing that b is uneven, and designating the sum of the series (iv) by $-2\log 2+2iS$, we have

$$S=i\sum_{n=1}^{n=\infty}\frac{(-1)^n}{n}\tan\left(\frac{na}{b}\pi\right)=\sum_{s=1}^{s=2b}\sum_{t=0}^{t=\infty}\frac{(-1)^t}{2bt+s}\tan\left(\frac{sa}{b}\pi\right)$$

$$=i\sum_{s=1}^{s=2b}\tan\left(\frac{sa}{b}\pi\right)\times\int_0^1\frac{x^{s-1}dx}{1-x^{2b}}.$$

Bnt if $\gamma=\varepsilon^{\frac{i\pi}{b}}$,

$$i\tan\left(\frac{sa\pi}{b}\right)=\frac{\gamma^{2as}-1}{\gamma^{2as}+1}=\sum_{\sigma=1}^{\sigma=b-1}(-1)^{\sigma-1}\gamma^{2a\sigma s},$$

and

$$\int_0^1\frac{x^{s-1}dx}{1-x^{2b}}=\frac{1}{2b}\sum_{\tau=1}^{\tau=2b}\gamma^{-\tau s}\int_0^1\frac{\gamma^\tau dx}{1-\gamma^\tau x};$$

whence

$$S=\frac{1}{2b}\sum_{\sigma=1}^{\sigma=b-1}(-1)^{\sigma-1}\sum_{\tau=1}^{\tau=2b}\int_0^1\frac{\gamma^\tau dx}{1-\gamma^\tau x}\sum_{s=1}^{s=2b}(-1)_s\gamma^{s(2a\sigma-\tau)}.$$

But

$$\sum_{s=1}^{s=2b}(-1)^s\gamma^{s(2a\sigma-\tau)}=2b \text{ or } 0,$$

according as the congruence $\tau \equiv 2a\sigma + b$, mod $2b$, is or is not satisfied. We have therefore

$$S = \sum_{\sigma=1}^{\sigma=b-1} (-1)^{\sigma} \int_0^1 \frac{\gamma^{2a\sigma} dx}{1+\gamma^{2a\sigma}x}$$

$$= \sum_{\sigma=1}^{\sigma=\frac{1}{2}(b-1)} (-1)^{\sigma} \int_0^1 \left(\frac{\gamma^{2a\sigma}}{1+\gamma^{2a\sigma}x} - \frac{\gamma^{-2a\sigma}}{1+\gamma^{-2a\sigma}x} \right) dx.$$

But
$$\int_0^1 \frac{\sin\theta\, dx}{1+2x\cos\theta+x^2} = [\tfrac{1}{2}\theta],$$

and hence, finally,

$$S = 2i\pi \sum_{\sigma=1}^{\sigma=\frac{1}{2}(b-1)} (-1)^{\sigma} \left[\sigma \frac{a}{b}\right]$$

$$= i\pi \sum_{\sigma=1}^{\sigma=b-1} (-1)^{\sigma} \left[\sigma \frac{a}{b}\right],$$

the symbol $\left[\sigma \frac{a}{b}\right]$ denoting the excess (positive or negative) of $\sigma \frac{a}{b}$ above the integral number lying absolutely nearest to $\sigma \frac{a}{b}$.

The method employed by Riemann to effect the preceding summation is due to Lejeune Dirichlet, and was employed by him in his memoir on the Arithmetical Progression (*Transactions of the Academy of Berlin for* 1837, sections 4 and 10), and in his "Recherches sur diverses applications de l'analyse infinitesimale à la théorie des nombres" (Crelle's *Journal*, vols. XIX. and XXI; see sections 1, 9 and 10 of the Memoir).

Again, denoting by S' the sum of the series

$$\sum_{n=1}^{n=\infty} \frac{(-1)^n}{n^2} \log\{4\cos^2\tfrac{1}{2}(n\theta)\},$$

we have, if $\theta = \frac{2a}{b}\pi$,

$$S' = 2\log 2 \times \sum \frac{(-1)^n}{n^2} \times \frac{1}{b^2} + \sum_{s=1}^{s=b-1} (-1)^s \log\left(4\cos^2 s\frac{a}{b}\pi\right) \sum_{t=0}^{t=\infty} \frac{(-1)^t}{(bt+s)^2}$$

$$= -\frac{\pi^2 \log 2}{6b^2} + \sum_{s=1}^{s=\frac{1}{2}(b-1)} (-1)^s \log\left(4\cos^2 s\frac{a}{b}\pi\right) \times \sum_{t=-\infty}^{t=+\infty} \frac{(-1)^t}{(bt+s)^2}.$$

But, by a formula due to Euler,

$$\sum_{t=-\infty}^{t=+\infty} \frac{(-1)^t}{(bt+s)^2} = \left[\frac{1}{s^2} + \frac{1}{(2b+s)^2} + \frac{1}{(2b-s)^2} + \frac{1}{(4b+s)^2} + \frac{1}{(4b-s)^2} + \ldots\right] - \left[\frac{1}{(b+s)^2} + \frac{1}{(b-s)^2} + \frac{1}{(3b+s)^2} + \frac{1}{(3b-s)^2} + \ldots\right]$$

$$= \frac{\pi^2}{4b^2}\operatorname{cosec}^2\left(\frac{s\pi}{2b}\right) - \frac{\pi^2}{4b^2}\sec^2\left(\frac{s\pi}{2b}\right)$$

$$= \frac{\pi^2}{b^2}\frac{\cos\left(\frac{s\pi}{b}\right)}{\sin^2\left(\frac{s\pi}{b}\right)};$$

whence, finally,

$$S' = -\frac{\pi^2}{6b^2}\log 2 + \frac{\pi^2}{b^2}\sum_{s=1}^{s=\frac{1}{2}(b-1)} (-1)^s \log\left(4\cos^2 s\frac{a}{b}\pi\right) \times \frac{\cos\left(\frac{s\pi}{b}\right)}{\sin^2\left(\frac{s\pi}{b}\right)}$$

$$= -\frac{\pi^2}{6b^2}\log 2 + \frac{\pi^2}{2b^2}\sum_{s=1}^{s=b-1} (-1)^s \log\left(4\cos^2 s\frac{a}{b}\pi\right) \times \frac{\cos\left(\frac{s\pi}{b}\right)}{\sin^2\left(\frac{s\pi}{b}\right)}.$$

When the series

$$\sum \frac{(-1)^n}{n^2} \log \{4 \cos^2 \tfrac{1}{2}(n\theta)\}$$

is divergent, we must regard the equation

$$4 \int_0^1 \sum \frac{(-1)^n}{n} \frac{\rho^n \varepsilon^{ni\theta}}{1+\rho^n \varepsilon^{ni\theta}} \times \frac{d\rho}{\rho}$$
$$= 2 \sum \frac{(-1)^n}{n^2} \log \{4 \cos^2 \tfrac{1}{2}(n\theta)\} + 4i \sum \frac{(-1)^n}{n^2} [\tfrac{1}{2}(n\theta)]$$

.....(ix),

as having no assignable meaning. When that series is convergent, the equation always has a meaning, but some explanation is required to show clearly what its meaning is. As to the signification of the left-hand member no question can arise. For just as by $\int_0^\infty \phi(x)\,dx$ we understand the limit of $\int_0^h \phi(x)\,dx$ when h is increased without limit, so by

$$\int_0^1 \sum \frac{(-1)^n}{n} \frac{\rho^n \varepsilon^{ni\theta}}{1+\rho^n \varepsilon^{ni\theta}} \times \frac{d\rho}{\rho}$$

we are to understand the limit of

$$\int_0^\rho \sum \frac{(-1)^n}{n} \frac{\rho^n \varepsilon^{ni\theta}}{1+\rho^n \varepsilon^{ni\theta}} \times \frac{d\rho}{\rho},$$

when $1-\rho$ is positive and decreases without limit. We thus have the equation

$$4 \int_0^1 \sum \frac{(-1)^n}{n} \frac{\rho^n \varepsilon^{ni\theta}}{1+\rho^n \varepsilon^{ni\theta}} \times \frac{d\rho}{\rho} = 4 \times \text{limit of}$$
$$\sum \frac{(-1)^n}{n^2} \log (1+\rho^n \varepsilon^{ni\theta}),$$

when $\rho = 1$. But before we can pass from this equation to the equation (ix), we have to show that the limit of

$$\sum \frac{(-1)^n}{n^2} \log (1+\rho^n \varepsilon^{ni\theta}),$$

when $\rho = 1$, is

$$\sum \frac{(-1)^n}{n^2} \log (1+\varepsilon^{ni\theta}).$$

And although this is true, it cannot be assumed without proof; any more than we could assume that the limit of the series

$$\tfrac{1}{2}(\sin x) + \tfrac{1}{3}(\sin 3x) + \tfrac{1}{5}(\sin 5x) + \dots,$$

when $x = 0$, is the same (which it certainly is not) as the value of that series when $x = 0$.

It would seem that Riemann in his fragmentary note intended to give the required proof; for he developes the expression

$$\sum \frac{(-1)^n}{n^2} \log(1 + q^n)$$

in a series proceeding by powers of q, (just as Jacobi has dealt with similar series in the *Fundamenta*) and finds

$$\sum \frac{(-1)^n}{n^2} \log(1 + q^n) = \sum \frac{1 + (-1)^n 2}{n} \Phi(n)\, q^n,$$

$\Phi(n)$ denoting the sum of the *uneven* divisors of n. He also enunciates and proves the following theorem (which is really due to Abel, Œuvres, Vol. I, p. 69).

"Si series

$$a_0 + a_1 + a_2 + \dots,$$

eo quo scripsimus ordine summata summam habet convergentem, functio ipsius r hâc serie

$$a_0 + a_1 r + a_2 r^2 + \dots$$

expressa, convergente r versus limitem 1, convergit versus valorem eundem."

But the fragment breaks off abruptly with the words "ex hoc theoremate.....facile deducitur," and it is not very easy to see how he proposed to complete the demonstration, because the series

$$\sum \frac{1 + (-1)^n 2}{n^2} \Phi(n)\, q^n$$

is certainly not always convergent when $q = \varepsilon^{i\vartheta}$. The following considerations may serve to replace Riemann's intended demonstration. Denoting $\varepsilon^{i\vartheta}$ by q_0, we have, in fact, to show that $\lim \sum \frac{(-1)^n}{n^2} \times \log \frac{1 + q^n}{1 + q_0^n} = 0$, when ρ

converges to unity, or $q=\rho\varepsilon^{i\theta}$ to $q_0=\varepsilon^{i\theta}$. The imaginary part of this series is

$$i\sum\frac{(-1)^n}{n^2}\lim\left[\tan^{-1}\left(\frac{\rho^n\sin n\theta}{1+\rho^n\cos n\theta}\right)-[\tfrac{1}{2}(n\theta)]\right]\ldots..(\text{x});$$

the real part is

$$\lim\sum\frac{(-1)^n}{n^2}\log[\rho^n+\{\tfrac{1}{2}(1-\rho^n)\}^2\sec^2\tfrac{1}{2}(n\theta)]\ldots\ldots(\text{xi});$$

it will suffice to consider the latter only. Let h be any very great number; and let Δ be so great that $4\Delta h$ is greater than the greatest value of $\sec^2\frac{1}{2}(n\theta)$ for any value of n not surpassing h (we observe that $\sec^2\frac{1}{2}(n\theta)$ cannot be infinite, because, in the cases under consideration, $\frac{\theta}{\pi}$ is not a rational fraction having an uneven numerator); also let ρ be so near to $+1$ that $1-\rho^n<\frac{1}{\Delta h}$ for all values of n not surpassing h; then, for all such values of n, $\rho^n+\{\frac{1}{2}(1-\rho^n)\}^2\sec^2\frac{1}{2}(n\theta)$ lies between $1-\frac{1}{\Delta h}$, and $1+\frac{1}{\Delta h}$; and, consequently,

$$\sum_{n=1}^{n=h}\frac{(-1)^n}{n^2}\log[\rho^n\times\{\tfrac{1}{2}(1-\rho^n)\}^2\sec^2\tfrac{1}{2}(n\theta)]\ldots\ldots(\text{xii})$$

is evanescent, being intermediate between the two values

$$\log\left(1\pm\frac{1}{\Delta h}\right)\times\sum_{n=1}^{n=h}\frac{1}{n^2},$$

which are both evanescent because $\frac{1}{\Delta h}$ is as small as we please.

Again, the series

$$\sum_{n=h+1}^{n=\infty}\frac{(-1)^n}{n^2}\log[\rho^n\times\{\tfrac{1}{2}(1-\rho^n)\}^2\sec^2\tfrac{1}{2}(n\theta)]\ldots\ldots(\text{xiii})$$

is evanescent, for $\rho^n+\{\frac{1}{2}(1-\rho^n)\}^2\sec^2\frac{1}{2}(n\theta)$ is certainly greater than $\frac{1}{4}$ and less than $\sec^2\frac{1}{2}(n\theta)$; whence this series is less in absolute magnitude than the sum of the two series

$$\log 4\times\sum_{n=h+1}^{n=\infty}\frac{1}{n^2},\quad\sum_{n=h+1}^{n=\infty}\frac{\log\sec^2\frac{1}{2}(n\theta)}{n^2};$$

and of these the first is evidently evanescent when h is very great; the second is also evanescent because, the series $\sum \frac{\log \sec^2 \frac{1}{2}(n\theta)}{n^2}$ being convergent in the cases here considered, the remainder after h terms vanishes when h is very great. Hence, finally, the series (xi), which is the sum of (xii) and (xiii), is evanescent.

ON FOURIER'S THEOREMS.

By *E. W. Hobson*, Christ's College, Cambridge.

I USE here the notation employed in Todhunter's *Integral Calculus* in the reproduction of Poisson's investigation of the series which represents a function between assigned limits. It is often felt as a difficulty that there is a certain arbitrariness in first making g indefinitely small and afterwards making α and β so. The following geometrical form of proof, which is, so far as I know, new, seems to elucidate the necessity for considering the limit of g as infinitely small compared with the limits of α and β.

Let P (fig. 1) be a point at a distance unity from the centre O of a circle of radius h.

Let θ be the angle which any radius OQ makes with a fixed radius OA. Let $f(\theta)$ be a function regarded as known at every point on the circumference between $\theta = 0$ and $\theta = \pi$. Let $POA = \theta'$.

We shall seek the value of the integral

$$\int_0^\pi \frac{1-h^2}{1-2h\cos(\theta-\theta')+h^2} f(\theta)\, d\theta.$$

Suppose PQ meets the circumference of the circle again in R, then the expression under the integral sign is equal to

$$\frac{PQ.PR}{PQ^2} f(\theta) = \frac{PR}{PQ} f(\theta).$$

Let QQ' be an element $h\,d\theta$ and let PO meet the circumference again in B.

Then $\angle QPQ' = \frac{QQ' \cos OQR}{PQ} = \frac{h\,d\theta}{PQ} \frac{PR-PQ}{2h}$

$$= \tfrac{1}{2}\left(\frac{PR}{PQ}\, d\theta - d\theta\right);$$

hence $$\frac{PR}{PQ}\,d\theta = 2\,(d\alpha + d\mu),$$

if $$d\alpha = \angle\, QBQ',\quad d\mu = \angle\, QPQ',$$

thus an element of the integral becomes

$$2f(\theta)\,(d\alpha + d\mu).$$

Now suppose h to become indefinitely near unity, that is, suppose P indefinitely near to the circle, then for all positions of Q at a finite distance from P we may consider P to be absolutely upon the circumference, in which case $\alpha + \mu = \frac{1}{2}\pi$, and therefore $d\alpha + d\mu = 0$ thus the corresponding element of the integral vanishes.

But if Q is not at a finite distance from P, since P is not absolutely upon the circumference of the circle, for such elements $d\alpha + d\mu$ is not zero. We must integrate for these separately and in order to include all such elements we must integrate along an arc which, though indefinitely small, is infinitely great compared with the distance of P from the circumference of the circle. We are then sure to have included all points in which $d\alpha + d\mu$ is not zero. This arc subtends an indefinitely small angle at B but an angle π at P, as it is when looked at from P an infinite straight line. Hence, if $f(\theta)$ is continuous for the value $\theta = \theta'$ the value of the integral is $2\pi f(\theta')$. If the function is discontinuous and changes from the value f_1 to f_2 at $\theta = \theta'$, then we must consider separately the arcs on either side of P, each of them subtends at P an angle $\frac{1}{2}\pi$; hence the integral is in that case $\pi\,(f_1 + f_2)$. If θ' is zero or π then we have to take only the one on one side of P, since the integration extends between π and 0; the value of the integral is then $\pi f(0)$ or $\pi f(\pi)$.

Let $$\theta = \frac{\pi v}{l},\quad \theta' = \frac{\pi x}{l},\quad f(\theta) = \phi(x).$$

Then since

$$\frac{1 - h^2}{1 - 2h\cos(\theta - \theta') + h^2} = 1 + \sum_1^\infty 2h^n \cos n(\theta - \theta'),$$

we have since $d\theta = \frac{\pi}{l}\,dv$ the theorem that

$$\frac{1}{2l}\int_0^l \phi(v)\,dv + \frac{1}{l}\sum_1^\infty \int_0^l \phi(v)\cos\frac{n\pi\,(v - x)}{l}\,dv,$$

is equal to $\phi(x)$ for any value of x between 0 and l for which the function is continuous, and equal to $\frac{1}{2}(\phi_1 + \phi_2)$ for a value of x for which the function changes from ϕ_1 to ϕ_2, and equal to $\frac{1}{2}\phi(0)$ or $\frac{1}{2}\phi(\pi)$ when $x = 0$ or π.

If we had integrated completely round the circle we should have obtained the theorem for the representation of the function between the values l and $-l$.

If we employ a straight line instead of a circle, we have a proof of the double integral theorem.

Let P (fig. 2) be a point at a distance h from a straight line. Let v be the distance of any point in the straight line from a fixed point O in it, and let a be the distance from O of the projection of P on the straight line. Consider a function $f(v)$. The angle $d\phi$ subtended by an element $\phi\phi'$ at P, is

$$\frac{k}{PQ} \cdot \frac{QQ'}{PQ} = \frac{k}{k^2 + (v-a)^2}\, dv;$$

hence (see Todhunter's *Integral Calculus*, Art. 12)

$$f(v)\, d\phi = f(v)\, dv \int_0^\infty e^{-ku} \cos u\,(a - v)\, du \ldots\ldots (1).$$

Now, let h be indefinitely diminshed, that is, let P be indefinitely mean to the straight line, then the angle which an element subtends at P vanishes except for elements about P; and as before we must take a portion of the straight line about P which is infinitely great compared with k, and this will subtend an angle π at P; hence, if we integrate the above equation (1) between the values p and q of v, we have the theorem that the limit when k is indefinitely diminished at the integral

$$\int_q^p f(v)\, dv \int_0^\infty e^{-ku} \cos u\,(a - v)\, du$$

is $\pi\phi(a)$ for values of a between p and q, for which $\phi(v)$ is continuous. Also for values at which it changes from ϕ_1 to ϕ_2 we have to take the elements on the two sides of P separately; hence, since each subtends an angle $\frac{1}{2}\pi$ at P, the above is equal to $\frac{1}{2}\pi(\phi_1 + \phi_2)$, and for $v = p$ or q it is $\frac{1}{2}\pi\phi(p)$ or $\frac{1}{2}\pi\phi(q)$.

A similar method applied to a sphere instead of a circle furnishes a proof of the expansion of a function in a series of Laplace's coefficients.

If QQ' (fig. 1) is an element of area of the sperical surface, the solid angle $d\varepsilon$ subtended by QQ' at P is

$$\frac{h^2 d\omega}{PQ^2} \cdot \frac{PR - PQ}{2h},$$

if $d\omega$ is the solid angle subtended by QQ' at the centre, this equals

$$\frac{hd\omega\,(1-h^2)}{2PQ^3} - \frac{hd\omega}{2PQ};$$

and since

$$\frac{1-h^2}{PQ^3} = \Sigma\,(2c+1)\,Q_c h^c,$$

where Q_c is the zonal harmonic at Q whose axis is OP, if $F(\mu\phi)$ be a function referred to any pole

$$\int \frac{1-h^2}{PQ^3} F(\mu\phi)\,d\mu d\phi = \Sigma\,(2c+1)\,h^c \int Q_c F(\mu\phi)\,d\mu d\phi.$$

The left-hand side of the equation equals

$$-\int \frac{d\omega}{PQ} F(\mu\phi) + 2\int \frac{d\varepsilon \,.\, F(\mu\phi)}{h}.$$

When P is indefinitely nearly coincident with p, each element at a finite distance vanishes, since it may easily be shewn that the solid angle subtended at p by QQ' is $\frac{1}{2}\frac{hd\omega}{AQ}$; but for the element about p,

$$\int \frac{hd\omega}{PQ} F(\mu\phi) = \int PQ d\varepsilon \,.\, F(\mu\phi) = 0,$$

and

$$\int d\varepsilon \,.\, F(\mu\phi) = 2\pi F(\mu'\phi'),$$

if μ' and ϕ' are the values of μ and ϕ at P; hence

$$4\pi F(\mu'\phi') = \Sigma\,(2c+1) \int Q_c F(\mu\phi)\,d\mu d\phi,$$

the ordinary theorem.

ON A SMITH'S PRIZE QUESTION, RELATING TO POTENTIALS.

By Professor *Cayley*.

A spherical shell is divided by a plane into two segments A and B, one of them so small that it may be regarded as a plane disk: trace the curves which exhibit the potentials of the two segments and of the whole shell respectively, in regard to a point P moving along the axis of symmetry of the two segments.

Criticise the following argument:

The potential of the segment A in regard to a point P, coordinates (x, y, z), *is one and the same function of* (x, y, z) *whatever be the position of P; similarly the potential of the segment B in regard to the same point P is one and the same function of* (x, y, z) *whatever be the position of P: hence the potential of the whole shell in regard to the point P is one and the same function of* (x, y, z) *whatever be the position of P.*

The question is taken from my memoir "On Prepotentials" *Phil. Trans.* vol. 165 (1875), pp. 675—774; and the figure of the curves is given p. 689. There is no difficulty in tracing them by means of the expression for the potential of a plane circular disk in regard to a point on its axis of symmetry: it was in order that they might be so traced, that one of the segments was taken to be small; but I had overlooked the circumstance that the formula for the disk is in fact only a particular case of a similar and equally simple formula for the spherical segment: viz. (as was found in one of the papers) the potential of a spherical segment in regard to a point on the axis is $= \frac{2\pi a}{\rho}(\rho_1 \sim \rho_2)$, where ρ, ρ_1, ρ_2 are the distances of the attracted point from the centre of the sphere and from the centre and circumference respectively of the segment. The segments might therefore just as well have been any two segments whatever, or (to take the most symmetrical case) they might have been hemispheres.

As to the argument: the assertion in regard to the potential of the segment A is based upon the consideration of this segment *alone*; and it is, on the ground that

we can without crossing the segment pass from any one position of P to any other position of P, inferred that the potential is one and the same function of the coordinates, whatever be the position of P: it is therefore unassailable by any considerations in relation to the non-existent segment B. Similarly the assertion in regard to the potential of the segment B is based upon the consideration of this segment alone, and it is unassailable upon any considerations in regard to the non-existent segment A: the potential of the whole sphere is certainly the sum of the potentials of the segments A and B: it is therefore altogether off the purpose to object that in the case of the whole sphere we cannot pass from a point outside the sphere to a point inside the sphere without crossing one or other of the segments A and B. I consider that the two assertions are each of them true, and that the conclusion a legitimate one, but it is true only in the sense in which $a+x+\sqrt{[(a-x)^2]}$ is one and the same function of x whatever be the value of x: this is so, if $\sqrt{[(a-x)^2]}$ denotes indifferently or successively the two functions $\pm(a-x)$, but if (a and x being real) $\sqrt{[(a-x)^2]}$ is taken to mean the positive value, then the function $a+x+\sqrt{[(a-x)^2]}$ is $=2a$ or $=2x$ according as $a-x$ is positive or negative.

In further illustration, let the dark line of fig. 3 represent the intersection of an unclosed surface, or segment, by the plane of xz taken to be that of the paper, and consider the potential of the segment in regard to a point P in the plane of the paper, coordinates x, z. We have the potential V defined as a function of x, z by an equation $V=$ definite integral depending on the parameters x, z, and being in general a transcendental function of (x, z); V is a real, one-valued, finite, continuous function of x, z: in particular if the point P moving in any manner transverses the dark line, there is not any discontinuity in the value of V. There is however in this case a discontinuity in the differential coefficients of V: if to fix the ideas we imagine P moving parallel to the axis of x, so that z is taken to be constant and V a function of x only, then when the path of P crosses the black line there is in general an abrupt change of value in $\frac{dV}{dx}$. Taking V as a coordinate y at right angles to the plane of the paper, a section by any plane parallel to that of xy, is (when the trace of the plane upon that of xz does not meet the dark line) a continuous curve; but when the trace meets the dark line, then for this value of x there is an abrupt change of direction in the section.

If (as may very well happen in particular cases) V is algebraically determinable, then (*quâ* one-valued function of (x, z)) V is not any root y at pleasure of an algebraical equation $\phi(x, y, z) = 0$, but it is for any given values of (x, z), some one determinate root y_1 of this equation: and we thus see how in this case the before-mentioned discontinuity in the value of $\frac{dV}{dx}$ must arise: viz. when the trace of the plane meets the dark line the section is a curve having a double point; and, for the positions of P on the two sides of the dark line, we have V the ordinate belonging to different branches of the curve of section. If the path of P passes through an extremity of the dark line then the curve of section will, instead of a double point, have in general a cusp; and when the path of P does not cross the dark line, then the curve of section is a continuous line without singularity. It may be added that the surface $\phi(x, y, z) = 0$ must have a nodal line which is as to a certain finite portion thereof crunodal, giving the before-mentioned double points of the sections, but as to the residue thereof acnodal or isolated.

It may happen that (the surface being algebraical) any particular section thereof instead of being a single curve having a double point as above breaks up into two distinct curves, so that for the two positions of P, we have V the ordinate of two distinct curves: and this is what really happens in the case of P a point on the axis of a circular disk, or a spherical segment: thus in the case of the disk, taking c for the radius, and x for the distance from the centre of the disk, the formula is $V = 2\pi\{\sqrt{(c^2 + x^2)} \pm x\}$; or writing $V \div 2\pi = y$, the section is made up of the two distinct hyperbolas $y(y - 2x) = c^2$, and $y(y + 2x) = c^2$.

It may be remarked that in each case, it is only for P on the axis that the potential is algebraical.

In the case of the hemispheres, drawing OM a radius at right angles to the axis, the formula for the potential of an axial point P is of the form $V = \frac{2\pi . OM}{OP}(PM \sim PA)$, or writing $V = 2\pi y$ we have for the hemisphere A, the curve (1) or (2) according as $(x - a)$ is positive or negative; and for the hemisphere B the curve (3) or (4) according as $x + a$ is positive or negative; viz. the equations are

$$(1) \quad y = \frac{a}{x}\{\sqrt{(a^2 + x^2)} - (x - a)\},$$

$$(2) \quad y = \frac{a}{x}\{\sqrt{(a^2+x^2)} + (x-a)\}.$$

$$(3) \quad y = \frac{a}{x}\{\sqrt{(a^2+x^2)} - (x+a)\}.$$

$$(4) \quad y = \frac{a}{x}\{\sqrt{(a^2+x^2)} + (x+a)\}.$$

being four cubic curves: the whole curve (1) is shown in fig. 4, and the others are equal or opposite curves: the rationalised equation of (1) is in fact

$$x = \frac{2a(y+a)}{(y+a)^2+a^2}.$$

and by writing $-a$ for a, and in each equation $-x$ for x, we have the rational equations of the other three curves.

But drawing only the required portions of the curves we have fig. 5 exhibiting the potentials of the two hemispheres A and B; and also the discontinuous potential of the whole shell, the ordinate for this last being the sum of the ordinates for the two hemispheres respectively.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, April 14th, Samuel Roberts, Esq., F.R.S., *President*, in the Chair. The Chairman feelingly alluded to the loss sustained by the Society in the recent death of Mr. T. Cotterill, M.A., formerly Fellow of St. John's College, Cambridge, who was for many years a Member of the Council, and had always taken a warm interest in the Society.

The following communications were made: "On the geodesic curvature of a curve on a surface," Prof. Cayley, F.R.S.; "On operative symbols in the differential calculus," Prof. Crofton, F.R.S.; "Note on the resolution into factors of numbers differing but slightly from π," Mr. J. W. L. Glaisher, F.R.S.; "On the nature of the quadric represented by the general equation of the second degree in tetrahedral coordinates and on the five focal quadrics of a cyclide," Mr. H. Hart, M.A.; "The discrimination of the maximum or minimum path of a ray of light reflected at a curve," Mr. H. M. Taylor, M.A.; "On certain tetrahedra specially related to four spheres meeting in a point" and "Historical note on Dr. Graves' theorem on confocal conics," The President.

Thursday, May 13th, S. Roberts, Esq., F.R.S., *President*, in the Chair. Prof. C. Niven, F.R.S., was admitted into the Society, and the following were elected Members: Mr. L. Rosenthal, B.A., Trinity College, Dublin, and Messrs. F.

Franklin, Ph.D., and C. A. van Velzer, and Miss Christine Ladd, the last three being members of the Johns Hopkins University, Baltimore.

The following papers were communicated: "On Ptolemy's theorem," Mr. C. W. Merrifield, F.R.S.; "The summation of certain hypergeometric series," Rev. T. R. Terry, M.A.; "Quaternion proof of Mr. S. Roberts' theorem of four co-intersecting spheres," Mr. J. J. Walker, M.A.; "Some solutions of the '15-girl' problem," Mr. E. Carpmael, M.A.; "Note on the coordinates of a tangent line to the curve of intersection of two quadrics," Mr. W. R. W. Roberts, M.A. Shorter communications were made by the President; Prof. Cayley, F.R.S.; Mr. Hart, M.A.; and Mr. J. J. Walker, M.A.

R. TUCKER, M.A., *Hon. Sec.*

ON THE FIRST RESOLVENT OF THE QUARTIC

$$y^4 + ay^2 + x_1 y - 1\tfrac{1}{2}\, a^2 = 0 \ldots\ldots\ldots\ldots\ldots (1).$$

By *Robert Rawson.*

1. DIFFERENTIATE (1) with respect to x, x_1 being a function of x and (a) being a constant;

therefore $$(4y^4 + 2ay^2 + x_1 y)\frac{dy}{dx} + y^2 \frac{dx_1}{dx} = 0 \ldots\ldots\ldots\ldots (2).$$

Eliminate y^4 in (2) by means of (1), then

$$(6ay^2 + 9x_1 y - a^2)\frac{dy}{dx} = 3y^2 \frac{dx_1}{dx} \ldots\ldots\ldots\ldots (3).$$

Put

$$(6ay + 9x_1 y - a^2)(ly^2 + my + n) = 3y^2 \frac{dx_1}{dx} \ldots\ldots (4).$$

Equation (4) is satisfied when l, m, n, are determined from the equations

$$(8a^3 + 27x_1^2)\ l + 6a\ \frac{dx_1}{dx} = 0,$$

$$(8a^3 + 27x_1^2)\, m - 9x_1 \frac{dx_1}{dx} = 0,$$

$$8a^3 + 27x_1^2)\ n\ + 3a^2 \frac{dx_1}{dx} = 0,$$

Substitute the above values of l, m, n in the equation

$ly^2+my+n=\frac{dy}{dx}$ and it becomes

$$\frac{8a^3+27x_1^2}{6a\frac{dx_1}{dx}}\cdot\frac{dy}{dx}+y^2-\frac{3x_1y}{2a}+\frac{a}{2}=0\ldots\ldots\ldots(5),$$

an equation which may be called the first resolvent of the quartic (1).

Each of the roots of this quartic is, therefore, a particular solution of (5).

2. A general solution of (5) is obtained as follows:

Let $y=y_1+v$ be a general solution...(6),

where y_1 is a root of the quartic (1), and v a new variable to be determined.

Substitute the above value of y in (5) and it becomes

$$\frac{8a^3+27x_1^2}{6a\frac{dx_1}{dx}}\left(\frac{dy_1}{dx}+\frac{dv}{dx}\right)+\left(y_1+v\right)^2-\frac{3x_1(y_1+v)}{2a}+\frac{a}{2}=0,$$

from which is derived

$$\frac{8a^3+27x_1^2}{6a\frac{dx_1}{dx}}\cdot\frac{dv}{dx}+\left(2y_1-\frac{3x_1}{2a}\right)v+v^2=0\ldots\ldots(7).$$

This equation is a well-known integrable form.

Integrate in the usual way, putting

$$(8a^3+27x_1^2)\,P=6a\frac{dx_1}{dx};$$

therefore
$$v=\frac{\exp\int\left\{\frac{3x_1}{2a}-2y_1\right)Pdx}{c+\int\left\{\exp\int\left(\frac{3x_1}{2a}-2y_1\right)Pdx\right\}Pdx}\ldots\ldots\ldots(8),$$

using the convenient notation of Prof. Cayley, viz. $\varepsilon^z=\exp z$.

Hence the general primitive or solution of (5) is

$$y_1+\frac{\exp\int\left(\frac{3x_1}{2a}-2y_1\right)Pdx}{C+\int\left\{\exp\int\left(\frac{3x_1}{2a}-2y_1\right)Pdx\right\}}\ldots\ldots\ldots(9).$$

Equation (5) is made linear of the second order by the assumption of

$$y = \frac{8a^3 + 27x_1^2}{6a\dfrac{dx_1}{dx}} \frac{du}{u\,dx} \quad \ldots\ldots\ldots\ldots\ldots(10);$$

substitute this value in (5) and it becomes

$$\frac{d^2u}{dx^2} + \left\{\frac{45x_1\dfrac{dx_1}{dx}}{8a^3 + 27x_1^2} - \frac{\dfrac{d^2x_1}{dx^2}}{\dfrac{dx_1}{dx}}\right\}\frac{du}{dx} + 18a^3\left\{\frac{\left(\dfrac{dx_1}{dx}\right)}{8a^3 + 27x_1^2}\right\}^2 u = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

The general solution of (11) is, therefore, the value of (u) determined from (10) wherein y takes the value given in equation (9).

4. The following particular case is of some interest in the theory of Elliptic Functions.

Put $x_1 = -4\left(x + \dfrac{1}{x}\right)$, and $a = -6$.

Then substitute these values in (1) and (5).

Equation (1) becomes

$$y^4 - 6y^2 - 4\left(x + \frac{1}{x}\right)y - 3 = 0 \quad \ldots\ldots\ldots\ldots(12);$$

and (5) becomes

$$3(1 - x^2)\frac{dy}{dx} - y^2 + \left(x + \frac{1}{x}\right)y + 3 = 0 \quad \ldots\ldots(13).$$

The various steps in the substitution and reduction being simple are omitted.

Equation (13) has been considered by Professor Cayley in the *Messenger of Mathematics*, vol. IV. pp. 69, 110, and *Elliptic Functions*, p. 248. And also by Mr. Hart in the *Messenger*, vol. IV. p. 125.

Its general solution is given by the substitution of $x_1 = -4\left(x + \dfrac{1}{x}\right)$, and $a = -6$ in equation (9) as follows:

$$y_1 + \frac{\left(\frac{x^2-1}{x}\right)^{\frac{1}{3}} \exp \frac{2}{3}\int \frac{y_1 dx}{1-x^2}}{C + \frac{1}{3}\int\left\{\left(\frac{1}{(x^2-1)^2 x}\right)^{\frac{1}{3}} \exp \frac{2}{3}\int \frac{y_1 dx}{1-x^2}\right\} dx} \quad \ldots(14).$$

5. Professor Cayley has pointed out that a particular solution is given by

$$y = v^2 x^{-\frac{1}{2}} + 2vx^{\frac{1}{4}} = v^{-2}x^{\frac{1}{2}} + 2v^{-1}x^{-\frac{1}{4}} \ldots\ldots\ldots(15).$$

Of course v is a root of the quartic

$$v^4 + 2x^{\frac{3}{4}}v^3 - 2x^{\frac{1}{4}}v - x = 0 \ldots\ldots\ldots\ldots(16).$$

And, to obtain a general solution, he transforms (13) into a differential equation of the second order from which the general solution is derived in the usual way. Such transformation is obviated by the procedure of Art. 2.

The elimination of v between (15) and (16) will give equation (12) as follows:

$$y^2 = (v^2 x^{-\frac{1}{2}} + 2vx^{\frac{1}{4}})(v^{-2}x^{\frac{1}{2}} + 2v^{-1}x^{-\frac{1}{4}}) = 2v^{-1}x^{\frac{3}{4}} + 2vx^{-\frac{3}{4}} + 5,$$

$$y^2 - 6 = 2v^{-1}x^{\frac{3}{4}} + 2vx^{-\frac{3}{4}} - 1;$$

therefore $y^4 - 6y^2 = 4v^{-2}x^{\frac{3}{2}} + 8v^{-1}x^{\frac{3}{4}} + 4v^2 x^{-\frac{3}{4}} + 8vx^{-\frac{3}{4}} + 3,$

$$-4\left(x + \frac{1}{x}\right)y = -4v^{-2}x^{\frac{3}{2}} - 8x^{-1}x^{\frac{3}{4}} - 4v^2 x^{-\frac{3}{4}} - 8vx^{-\frac{3}{4}}.$$

Then $$y^4 - 6y^2 - 4\left(x + \frac{1}{x}\right)y - 3 = 0,$$

which agrees with equation (12), as it should do.

6. A root of (12) may be found by changing the independent variable x to another independent variable z connected with the former variable by a given relation.

This has been done by Professor Cayley (see *Messenger*, vol. IV. p. 111). Put $y = \phi(z)$;

therefore $$x + \frac{1}{x} = \frac{\phi(z)^2\{\phi(z)^2 - 6\} - 3}{4\phi(z)} \ldots\ldots\ldots(17).$$

If $$\phi(z)^2 = \frac{2z^2 + 5z + 2}{z},$$

therefore $$x + \frac{1}{x} = \frac{z^4 + 2z^3 + 2z + 1}{z\sqrt{z}\sqrt{(2z^2 + 5z + 2)}}.$$

From which

$$x^2 = z^3 \frac{2 + z}{1 + 2z} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18),$$

which agrees with Professor Cayley's result.

General values of x can only be obtained by the solution of the quartic $z^4 + 2z^3 - 2x^2z - x^2 = 0$. The selection seems to be between this quartic and the quartic (12). This remark applies equally well, if I mistake not, to Mr. Hart's solution of the same equation.

SOLUTION OF A SENATE-HOUSE PROBLEM.

By Professor *Cayley*.

PROVE that if $a + b + c = 0$ and $x + y + z = 0$, then

$$\begin{aligned} &4(ax + by + cz)^3 \\ &- 3(ax + by + cz)(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \\ &- 2(b - c)(c - a)(a - b)(y - z)(z - x)(x - y) \\ &- 54abcxyz = 0. \end{aligned}$$

I do not know the origin of this identity, nor do I see any very simple way of proving it: that which seems the most straightforward way is to transform the third line, which, omitting the factor -2, is

$$\begin{vmatrix} 1, & 1, & 1 \\ a, & b, & c \\ a^2, & b^2, & c^2 \end{vmatrix} \cdot \begin{vmatrix} 1, & 1, & 1 \\ x, & y, & z, \\ x^2, & y^2, & z^2 \end{vmatrix},$$

$$= \begin{vmatrix} 3, & a + b + c, & a^2 + b^2 + c^2 \\ x + y + z, & ax + by + cz, & a^2x + b^2y + c^2z \\ x^2 + y^2 + z^2, & ax^2 + by^2 + cz^2, & a^2x^2 + b^2y^2 + c^2z^2 \end{vmatrix};$$

and therefore when $a+b+0$ and $x+y+z=0$, is

$$\begin{aligned}&=3\,(ax+by+cz)\,(a^2x^2+b^2y^2+c^2z^2)\\&-3\,(a^2x+b^2y+c^2z)\,(ax^2+by^2+cz^2)\\&-\,(ax+by+cz)\,(a^3+b^2+c^2)\,(x^2+y^2+z^2);\end{aligned}$$

or as this may be written

$$\begin{aligned}&=6\,(ax+by+cz)\,(a^2x^2+b^2y^2+c^2z^2)\\&-\,(ax+by+cz)\,(a^2+b^2+c^2)\,(x^2+y^2+z^2)\\&\left.\begin{aligned}&-3\,(ax+by+cz)\,(a^2x^2+b^2y^2+c^2z^2)\\&-3\,(a^2x+b^2y+c^2z)\,(ax^2+by^2+cz^2)\end{aligned}\right\},\end{aligned}$$

and here the third and fourth lines, omitting the factor -3, are

$$2\,(a^3x^3+b^3y^3+c^3z^3)$$

$$+(ab^2+a^2b)(xy^2+x^2y)+(ac^2+a^2c)(xz^2+x^2z)+(bc^2+b^2c)(yz^2+y^2z),$$

where in the second line, in virtue of the two relations, each of the three terms is $=abcxyz$, and the whole is thus

$$\begin{aligned}&=\;\;2\,(a^3x^3+b^3y^3+c^3z^3)\\&\quad+3abcxyz.\end{aligned}$$

The product of the two determinants is thus

$$\begin{aligned}&=\;\;6\,(ax+by+cz)\,(a^2x^2+b^2y^2+c^2z^2)\\&\quad-\,(ax+by+cz)\,(a^2+b^2+c^2)\,(x^2+y^2+z^2)\\&\quad-6\,(a^3x^3+b^3y^3+c^3z^3)\\&\quad-9\;abcxyz;\end{aligned}$$

and this being so the identity to be verified is

$$\begin{aligned}&\qquad\qquad 4\;\;(ax+by+cz)^3\\&+(-3+2=)-1\;\;(ax+by+cz)\,(a^2+b^2+c^2)\,(x^2+y^2+z^2)\\&\qquad\qquad-12\,(ax+by+cz)\,(a^2x^2+b^2y^2+c^2z^2)\\&\qquad\qquad+12\,(a^3x^3+b^3y^3+c^3z^3)\\&+(18-54=)-36abcxyz\qquad=0.\end{aligned}$$

We have here the terms

$$12\,(a^3x^3+b^3y^3+c^3z^3-3abcxyz),$$

$$=12\,(ax+by+cz)\,(a^2x^2+b^2y^2+c^2z^2-bcyz-cazx-abxy),$$

so that the left hand side is now divisible by $ax+by+cz$, and throwing out this factor the equation becomes

$$\begin{aligned}&4\,(ax+by+cz)^2\\ -\;&(a^2+b^2+c^2)\,(x^2+y^2+z^2)\\ &-12\,(a^2x^2+b^2y^2+c^2z^2)\\ &+12\,(a^2x^2+b^2y^2+c^2z^2-bcyz-cazx-abxy)=0;\end{aligned}$$

or, as this may be written,

$$\begin{aligned}&4\,(a^2x^2+b^2y^2+c^2z^2-bcyz-cazx-abxy)\\ -\;&(a^2+b^2+c^2)\,(x^2+y^2+z^2)=0,\end{aligned}$$

which under the assumed relations $a+b+c=0$, $x+y+z=0$ may be verified without difficulty. It may be remarked that we have identically

$$\begin{aligned}&8\,(a^2x^2+b^2y^2+c^2z^2-bcyz-cazx-abxy)\\ &-2\,(a^2+b^2+c^2)\,(x^2+y^2+z^2)\\ =(x+y+z)&\left\{\begin{array}{l}x\,(\;3a^2-\;b^2-\;c^2+2bc-2ca-2ab)\\ +y\,(-\;a^2+3b^2-\;c^2-2bc+2ca-2ab)\\ +z\,(-\;a^2-\;b^2+3c^2-2bc-2ca+2ab)\end{array}\right\}\\ +(a+b+c)&\left\{\begin{array}{l}a\,(\;3x^2-\;y^2-\;z^2+2yz-2zx-2xy)\\ +b\,(-\;x^2+\;y^2-\;z^2-2yz+2zx-2xy)\\ +c\,(-\;x^2-\;y^2+3z^2-2yz-2zx+2xy)\end{array}\right\},\end{aligned}$$

which is a more complete form of the last-mentioned theorem.

ON CERTAIN MATHEMATICAL IDENTITIES OCCURRING IN THOMSON AND TAIT'S "NATURAL PHILOSOPHY."

By *Frederick Purser, M.A.*

In their discussion (*Natural Philosophy*, pp. 541, 542) of the torsion of a rectangular prism, Thomson and Tait call attention to the fact that a slight variation in the mode of treatment leads to results apparently different but necessarily identical and add the comment, that "The comparison of the results gives astonishing theorems of pure mathematics such as rarely fall to the lot of those mathematicians who confine themselves to pure analysis or geometry instead of allowing themselves to be led into the rich and beautiful fields of mathematical truth which lie in the way of physical research." Such a remark coming from such writers naturally stimulates curiosity and leads us to enquire for the mathematical ground on which these theorems, though suggested by physical investigations, must after all really rest.

On examination, it appears that the clue to the remarkable identities in question, is to be found in the well-known transformation in elliptic functions from u, k, K to u, k', K'.

To show this is the object of the present note.

The identities are the following (see *Natural Philosophy*, p. 542, Nos. 45, 46):

(I).

$$-\tau xy+\frac{8\tau}{\pi^3}b^2\sum\frac{(-1)^r}{(2r+1)^3}\frac{e^{(2r+1)\frac{\pi x}{b}}-e^{-(2r+1)\frac{\pi x}{b}}}{e^{(2r+1)\frac{\pi a}{2b}}+e^{-(2r+1)\frac{\pi a}{2b}}}\sin(2r+1)\frac{\pi y}{b}$$

$$=\tau xy-\frac{8\tau}{\pi^3}a^2\sum\frac{(-1)^r}{(2r+1)^3}\frac{e^{(2r+1)\frac{\pi y}{a}}-e^{-(2r+1)\frac{\pi y}{a}}}{e^{(2r+1)\frac{\pi b}{2a}}+e^{-(2r+1)\frac{\pi b}{2a}}}\sin(2r+1)\frac{\pi x}{a},$$

these being equivalent expressions for the torsion function γ.

$$(\text{II}).^{*}\quad nab^3\left\{\tfrac{1}{3}-\frac{2^6}{\pi^5}\frac{b}{a}\sum\frac{1}{(2r+1)^5}\frac{1-e^{-(2r+1)\frac{\pi a}{b}}}{1+e^{-(2r+1)\frac{\pi a}{b}}}\right\}$$

$$=nba^3\left\{\tfrac{1}{3}-\frac{2^6}{\pi^5}\frac{a}{b}\sum\frac{1}{(2r+1)^5}\frac{1-e^{-(2r+1)\frac{\pi b}{a}}}{1+e^{-(2r+1)\frac{\pi b}{a}}}\right\},$$

both sides representing the torsional rigidity N.

Let now $$\phi(u)=\frac{\mathrm{sn}(u,k)}{\mathrm{dn}(u,k)}.$$

Starting from the known expansion

$$\phi(u)=\frac{2\pi}{kk'K}\Sigma(-1)^r\frac{1}{q^{\frac{1}{2}(2r+1)}+q^{-\frac{1}{2}(2r+1)}}\sin\frac{(2r+1)\pi u}{2K},$$

substituting $X+iY$, $X-iY$ for u, and adding, we have, writing for brevity,

$$M=\frac{2\pi}{kk'K},\quad P_r=(-1)^r\frac{1}{q^{\frac{1}{2}(2r+1)}+q^{-\frac{1}{2}(2r+1)}},$$

$$\phi(X+iY)+\phi(X-iY)=2M\Sigma P_r\sin\frac{(2r+1)\pi X}{2K}\cosh\frac{(2r+1)\pi Y}{2K}.$$

Now $$\phi(X+iY)=\phi i(Y-iX),$$

$$\phi(X-iY)=\phi-i(Y+iX)=-\phi i(Y+iX).$$

Also by the theory of elliptic functions

$$\phi(iu,k)=i\phi(u,k').$$

If then $$\psi(u)=\frac{\mathrm{sn}(u,k')}{\mathrm{dn}(u,k')}$$

we have

$$\phi(X+iY)+\phi(X-iY)=i\{\psi(Y-iX)-\psi(Y+iX)\}.$$

Let now by transformation to complementary modulus

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* I have replaced the $\frac{2^6}{\pi^5}$ of the book by $\frac{2^5}{\pi^6}$. This correction is eaily seen to be necessary, by taking $\frac{b}{a}$ evanescent and comparing coefficients of a^3b.

M, P_r become M', P_r', then expanding $\psi(u)$ and substituting, we find

$$i\psi(Y-iX)-i\psi(Y+iX)$$
$$=2M'\Sigma P_r'\cos\frac{(2r+1)\pi Y}{2K'}\sinh\frac{(2r+1)\pi X}{2K'}.$$

We have thus the identity

$$M\Sigma P_r\sin\frac{(2r+1)\pi X}{2K}\cosh\frac{(2r+1)\pi Y}{2K}$$
$$=M'\Sigma P_r'\cos\frac{(2r+1)\pi Y}{2K'}\sinh\frac{(2r+1)\pi X}{2K'}.$$

If now we write

$$Y=\frac{2K'y}{b},\quad X=\frac{2K'x}{b},\quad \frac{K}{K'}=\frac{a}{b}.$$

This identity becomes

$$b\Sigma(-1)^r\sin\frac{(2r+1)\pi x}{a}\frac{\cosh\frac{(2r+1)\pi y}{a}}{\cosh\frac{(2r+1)\pi b}{2a}}$$
$$=a\Sigma(-1)^r\cos\frac{(2r+1)\pi y}{b}\frac{\sinh\frac{(2r+1)\pi x}{b}}{\cosh\frac{(2r+1)\pi a}{2b}}\ldots\ldots(A).$$

From the fundamental identity (A), the required formulæ (I), (II) may now be deduced by integration.

Integrating with respect to y, we have

$$\Sigma(-1)^r\sin\frac{(2r+1)\pi x}{a}\frac{1}{2r+1}\frac{\sinh\frac{(2r+1)\pi y}{a}}{\cosh\frac{(2r+1)\pi b}{2a}}$$
$$=\Sigma(-1)^r\sin\frac{(2r+1)\pi y}{b}\frac{1}{2r+1}\frac{\sinh\frac{(2r+1)\pi x}{b}}{\cosh\frac{(2r+1)\pi a}{2b}}\ldots(B),$$

no supplementary function of x being added since both sides vanish for $y=0$.

Integrating a second time

$$\frac{a}{\pi}\Sigma(-1)^r\frac{1}{(2r+1)^2}\sin\frac{(2r+1)\pi x}{a}\frac{\cosh\dfrac{(2r+1)\pi y}{a}}{\cosh\dfrac{(2r+1)\pi b}{2a}}$$

$$=-\frac{b}{\pi}\Sigma(-1)^r\frac{1}{(2r+1)^2}\cos\frac{(2r+1)\pi y}{b}\frac{\sinh\dfrac{(2r+1)\pi x}{b}}{\cosh\dfrac{(2r+1)\pi a}{2b}}+\frac{\pi x}{4}$$

.....(C).

The supplementary function $\frac{\pi x}{4}$ being determined by putting $y=\frac{1}{2}b$, and applying the series

$$\frac{\pi\theta}{4}=\sin\theta-\frac{1}{3^2}\sin 3\theta+\text{----}$$

θ varying from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$.

A third integration now gives

$$\frac{a^2}{\pi^2}\Sigma(-1)^r\frac{1}{(2r+1)^3}\sin\frac{(2r+1)\pi x}{a}\frac{\sinh\dfrac{(2r+1)\pi y}{a}}{\cosh\dfrac{(2r+1)\pi b}{2a}}$$

$$=-\frac{b^2}{\pi^2}\Sigma(-1)^r\frac{1}{(2r+1)^3}\sin\frac{(2r+1)\pi y}{b}\frac{\sinh\dfrac{(2r+1)\pi x}{b}}{\cosh\dfrac{(2r+1)\pi a}{2b}}+\frac{\pi xy}{4}$$

.....(D).

This may now be written

$$-xy+\frac{8}{\pi^3}b^2\Sigma(-1)^r\frac{1}{(2r+1)^3}\sin\frac{(2r+1)\pi y}{b}\frac{\sinh\dfrac{(2r+1)\pi x}{b}}{\cosh\dfrac{(2r+1)\pi a}{2b}}$$

$$=xy-\frac{8}{\pi^3}b^2\Sigma(-1)^r\frac{1}{(2r+1)^3}\sin\frac{(2r+1)\pi x}{a}\frac{\sinh\dfrac{(2r+1)\pi y}{a}}{\cosh\dfrac{(2r+1)\pi b}{2a}},$$

which is the required identity I.

The identity (II) is obtained by two further integrations, employing now for determination of supplementary functions of x the Fourierian series

$$\frac{\pi}{8}\left(\frac{\pi^2\theta}{4}-\frac{\theta^3}{3}\right)=\sin\theta-\frac{1}{3^4}\sin 3\theta+\dots,$$

θ as before ranging from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$.

We have thus

$$\frac{a^3}{\pi^3}\Sigma(-1)^r\frac{1}{(2r+1)^4}\sin\frac{(2r+1)\pi x}{a}\frac{\cosh\dfrac{(2r+1)\pi y}{a}}{\cosh\dfrac{(2r+1)\pi b}{2a}}$$

$$=\frac{b^3}{\pi^3}\Sigma(-1)^r\frac{1}{(2r+1)^4}\cos\frac{(2r+1)\pi y}{b}\frac{\sinh\dfrac{(2r+1)\pi x}{b}}{\cosh\dfrac{(2r+1)\pi a}{2b}}$$

$$+\frac{\pi x}{8}\left(y^2-\frac{b^2}{4}\right)+\frac{\pi a^3}{8}\left(\frac{x}{4a}-\frac{x^3}{3a^3}\right)\dots\dots\dots\dots\dots\dots(E),$$

$$\frac{a^4}{\pi^4}\Sigma(-1)^r\frac{1}{(2r+1)^5}\sin\frac{(2r+1)\pi x}{a}\frac{\sinh\dfrac{(2r+1)\pi y}{a}}{\cosh\dfrac{(2r+1)\pi b}{2a}}$$

$$=\frac{b^4}{\pi^4}\Sigma(-1)^r\frac{1}{(2r+1)^5}\sin\frac{(2r+1)\pi y}{b}\frac{\sinh\dfrac{(2r+1)\pi x}{b}}{\cosh\dfrac{(2r+1)\pi a}{2b}}$$

$$+\pi xy\left(\frac{y^2-x^2}{24}+\frac{a^2-b^2}{32}\right)\dots\dots\dots\dots\dots\dots(F).$$

Writing now in (F) $x=\frac{1}{2}a$, $y=\frac{1}{2}b$, we find

$$\frac{a^4}{\pi^5}\sum\frac{1}{(2r+1)^5}\tanh\frac{(2r+1)\pi b}{2a}$$

$$=\frac{b^4}{\pi^5}\sum\frac{1}{(2r+1)^5}\tanh\frac{(2r+1)\pi a}{2b}+\frac{ab\,(a^2-b^2)}{192},$$

or
$$ab^3\left\{\tfrac{1}{3}-\frac{2^6}{\pi^5}\frac{b}{a}\sum\frac{1}{(2r+1)^5}\,\frac{1-e^{-(2r+1)\frac{\pi a}{b}}}{1+e^{-(2r+1)\frac{\pi a}{b}}}\right\}$$

$$=ba^3\left\{\tfrac{1}{3}-\frac{2^6}{\pi^5}\frac{a}{b}\sum\frac{1}{(2r+1)^5}\,\frac{1-e^{-(2r+1)\frac{\pi b}{a}}}{1+e^{-(2r+1)\frac{\pi b}{a}}}\right\},$$

the identity (II).

I may add the following remarks:

(1). The identical series (B) obtained by integration of (A) with respect to q will be found by integration of the original elliptic formula to represent the function

$$\tfrac{1}{4}\log\frac{1+\mathrm{cn}\,(K-X,\,k)\,\mathrm{cn}\,(K'-Y,\,k')}{1-\mathrm{cn}\,(K-X,\,k)\,\mathrm{cn}\,(K'-Y,\,k')},$$

or $\tfrac{1}{4}\log\dfrac{1+\mathrm{cn}\,\lambda\,(\frac{1}{2}a-x,\,k)\,\mathrm{cn}\,\{\lambda(\frac{1}{2}b-y),\,k'\}}{1-\mathrm{cn}\,\lambda\,(\frac{1}{2}a-x,\,k)\,\mathrm{cn}\,\{\lambda(\frac{1}{2}b-y),\,k'\}}$, $\lambda=\dfrac{2K}{a}=\dfrac{2K'}{b}$.

If we had integrated (A) with respect to x we should have obtained series representing the conjugate function

$$\tfrac{1}{2}\tan^{-1}\frac{k}{k'}\,\frac{\mathrm{cn}\,(\lambda x,\,k)}{\mathrm{cn}\,(\lambda y,\,k')},$$

a function which I showed in a previous paper (No. 69, 1877), represented, under certain conditions, the permanant distribution of temperature in a rectangular lamina.

A series proportional to this latter function would also be obtained by differentiating twice the function v (see Thomson and Tait, p. 542, No. 44) with respect to y. It is readily seen that this ought to be the case. For v is determined as satisfying $\dfrac{d^2v}{dx^2}+\dfrac{d^2v}{dy^2}=0$, and assuming the values $-\tau\,(\frac{1}{4}b^2-y^2)$ for $x=\pm\frac{1}{2}a$, 0 for $y=\pm\frac{1}{2}b$. It is then easily seen that $\dfrac{d^2v}{dy^2}$ satisfies the same differential equation and assumes the values 2τ for $x=\pm\frac{1}{2}a$, while from its trigonometrical form it vanishes for $y=\pm\frac{1}{2}b$. It is therefore a potential satisfying conditions of the same form as that in the temperature problem.

(2). With respect, in general, to the employment of imaginary values for u in the Fourierian series for the elliptic functions, I would note an inadvertent statement in the *Théorie des Intégrales et des Fonctions Elliptiques*, translated by Graindorge from the German of Schloemilch (p. 188), to the effect that for such values these series cease to be convergent. It is easily shown that this is not necessarily the case. Consider, in fact, as a specimen, the series for $\operatorname{sn} u$, in which the general term A_n is proportional to $\frac{q^{\frac{1}{2}(2n+1)}}{1-q^{2n+1}} \sin \frac{(2n+1)\pi u}{2K}$.

If in this we write for u, $x+iy$, the modulus of the term will become

$$\tfrac{1}{2} \frac{q^{\frac{1}{2}(2n+1)}}{1-q^{2n+1}} \sqrt{(e^{2\eta} + e^{-2\eta} - 2\cos 2\xi)},$$

where $$\eta = \frac{(2n+1)\pi y}{2K}, \quad \xi = \frac{(2n+1)\pi x}{2K}.$$

Now as n increases this will remain $< \frac{1}{2}\kappa . q^{\frac{1}{2}(2n+1)} e_\eta$, κ being a fixed finite quantity; or $\operatorname{mod} A_n < \kappa \left(e^{\frac{\pi(y-K')}{-K}}\right)^{\frac{1}{2}(2n+1)}$. It appears then that the series is convergent so long as $y < K'$. The same reasoning holds for the other leading elliptic series including that for $\frac{\operatorname{sn} u}{\operatorname{dn} u}$ employed in the present paper. The restriction above shows that the identities are only proved for values of y between $-\frac{1}{2}b$, $\frac{1}{2}b$ corresponding also to the range within which the trigonometric series determining supplementary functions were valid.

This remark might, I believe, be extended, and we might safely say that, in the case of all the Fourierian series in Elliptic Functions, when imaginary values of the variable are used, the limits of mathematical convergence coincide with those assigned by the physical problem.

ON THE CALCULUS OF FINITE DIFFERENCES.

By *W. H. L. Russell, F.R.S.*

THE following theorems relative to this subject will, I hope, be found interesting. I commence with some results in general finite differentiation.

1. We shall have

$$\Delta^n \frac{1}{x} = (-1)^n \frac{1.2.3\ldots n}{x\,(x+1)\,(x+2)\ldots(x+n)},$$

$$\Delta^n \frac{1}{x^2} = (-1)^n \frac{1.2.3\ldots(n+1)\,x^n + A_1^{(n)} x^{n-1} + A_2^{(n)} x^{n-2} + \ldots}{x^2\,(x+1)^2\,(x+2)^2\ldots(x+n)^2},$$

where $A_1^{(n)}$, $A_2^{(n)}$ are found by linear equations of differences of the first order, obtained by substituting the above expression in the equation

$$\Delta^{n+1} \frac{1}{x^2} = \Delta^n \frac{1}{(x+1)^2} - \Delta^n \frac{1}{x^2},$$

and equating coefficients; thus we have

$$A_1^{(n+1)} - (n+3)\,A_1^{(n)} = 1.2.3\ldots(n+1)\,\tfrac{1}{2}\,(n^2+5n+2);$$

whence $$A_1^{(n)} = 1.2.3\ldots(n+2)\,\Sigma \frac{n^2+5n+2}{2\,(n+2)\,(n+3)};$$

$A_1^{(n)}$ being found, we have another linear equation of the first order to determine $A_2^{(n)}$, and so we proceed. It is seen at once that

$$A_n^{(n)} = 1^2.2^2.3^2\ldots n^2.$$

Also $$\Delta^n \frac{1}{x^3} = (-1)^n \frac{3.4\ldots(n+2)\,x^{2n} + A_1^{n} x^{2n-1} + \ldots}{x^3\,(x+1)^3\,(x+2)^3\ldots(x+n)^3},$$

where $A_1^{(n)}$, $A_2^{(n)}$ are determined as before, and

$$A_{2n}^{(n)} = 1^3.2^3.3^3\ldots(n+1)^3.$$

By an analogous process $\Delta^n \Gamma(x) =$

$$\left\{x^n + n\,(n-3)\,\frac{x^{n-1}}{1.2} + (3n^4 - 22n^3 + 57n^2 - 38n)\,\frac{x^{n-2}}{1.2.3.4} + \ldots\right\} \Gamma x,$$

with expressions of like nature for

$$\Delta^n \frac{1}{\Gamma x}, \quad \Delta^n \Gamma(x)^2, \text{ and } \Delta^n \frac{1}{\Gamma(x)^2},$$

and we shall have

$$\Delta^n \frac{1}{a^x+1} = \frac{(-1)^n (a-1)^n (a^{\frac{n(n-1)}{2}} a^{nx} + A a^{(n-1)x} + \ldots)}{(a^{x+n}+1)(a^{x+n-1}+1)\ldots(a^x+1)},$$

from this expression we may find $\Delta^n \tan x$, $\Delta^n \frac{1}{\sin x}$, &c. It will be seen that my chief object in this investigation has been to find the first terms in the numerators of resulting fractions. When these are known, the other terms are ascertained without difficulty.

2. Let us now endeavour to ascertain the condition that a rational fraction may be integrated in finite terms. The result will of course be another rational fraction. Let the given fraction and its integral be $\frac{\phi(x)}{\chi(x)}$ and $\frac{\phi_1(x)}{\chi_1(x)}$, then

$$\chi_1(x) = (x-a_1)(x-a_2)(x-a_3)\ldots,$$

then
$$\frac{\phi_1(x)}{\chi_1(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \frac{A_3}{x-a_3} + \ldots,$$

when A_1, A_2, $A_3\ldots$ do not contain x.

Hence

$$\frac{\phi(x)}{\chi(x)} = \frac{A_1}{x-a_1+1} - \frac{A_1}{x-a_1} + \frac{A_2}{x-a_3+1} - \frac{A_1}{x-a_2} + \ldots,$$

whence we have

$$\chi(x) = (x-a_1)(x-a_1+1)(x-a_2)(x-a_2+1)\ldots,$$

$$\frac{\phi(a_1)}{\chi'(a_1)} + \frac{\phi(a_1-1)}{\chi'(a_1-1)} = 0, \quad \frac{\phi(a_2)}{\chi'(a_2)} + \frac{\phi(a_2-1)}{\chi'(a_2-1)} = 0,$$

as the necessary conditions that the proposed fraction may be integrable.

There are however certain cases in which these conditions require to be modified.

Let $\chi_2(x) = (x-a_1)(x-a_1+1)(x-a_1+2)(x-a_2)\ldots,$

then
$$\frac{\phi_1(x)}{\chi_1(x)} = \frac{A_1}{x-a_1} + \frac{A_1'}{x-a_1+1} + \frac{A_1''}{x-a_2+2} + \ldots,$$

then $$\frac{\phi(x)}{\chi(x)} = \frac{A_1}{x-a_1+1} - \frac{A_1}{x-a_1}$$

$$+ \frac{A_1'}{x-a_1+2} - \frac{A_1'}{x-a_1+1} + \frac{A_1''}{x-a_2+3} - \frac{A_1''}{x-a_2+2} + \dots.$$

Hence $\chi(x) = (x-a_1)(x-a_1+1)(x-a_1+2)(x-a_1+3)\dots,$

$$A_1 - A_1' = \frac{\phi(a_1-1)}{\chi'(a_1-1)}, \quad A_1' - A_1'' = \frac{\phi(a_1-2)}{\chi'(a_1-2)}.$$

Hence, we have, as the modified condition,

$$\frac{\phi(a_1)}{\chi'(a_1)} + \frac{\phi(a_1-1)}{\chi'(a_1-1)} + \frac{\phi(a_1-2)}{\chi'(a_1-2)} + \frac{\phi(a_1-3)}{\chi'(a_1-3)} = 0.$$

The same method manifestly applies when any number of the factors of the denominator successively differ by unity. Suppose the denominator of the fraction proposed for integration to be of the form

$$(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5),$$

then it is manifest that we must have either

$$a_2 = a_1 - 1, \quad a_3 = a_1 - 2, \quad a_5 = a - 1,$$

or $$a_2 = a_1 - 1, \quad a_3 = a_1 - 2, \quad a_4 = a_1 - 3, \quad a_5 = a_1 - 4.$$

When these are equal factors in the denominator this method requires very slight alteration. The innumerable applications of these principles to the summation of series I leave to the reader.

3. The finite integration of irrational functions according to the methods of the calculus of finite differences offers, so far as I have observed, but few interesting results. I therefore proceed to the exponential functions. We shall have

$$\Sigma \frac{\sin(x+n)\theta}{\cos x\theta \cos(x+1)\theta \dots \cos(x+2n)\theta}$$

$$= \frac{1}{2\sin n\theta} \frac{1}{\cos x\theta \dots \cos(x+2n-1)\theta},$$

$$\Sigma \frac{\sin 2(x+n)\theta}{\cos^2 x\theta \cos^2(x+1)\theta \dots \cos^2(x+2n)\theta}$$

$$= \frac{1}{\sin 2n\theta} \frac{1}{\cos^2 x\theta \cos^2(x+1)\theta \dots \cos^2(x+2n-1)\theta},$$

$$\Sigma \frac{\sin 3(x+n)\theta \sin 3\theta + 3\sin(x+n)\theta \sin\theta}{\cos^3 x\theta \cos^3(x+1)\theta \ldots \cos^3(x+2n)\theta}$$

$$= \frac{4}{\cos^3 x\theta \cos^3(x+1) \ldots \cos^3(x+2n-1)\theta},$$

$$\Sigma \frac{1}{\cos x\theta \operatorname{cotan}(x+1)\theta \ldots \operatorname{cotan}(x+n)\theta \cos(x+n+1)\theta}$$

$$= \frac{1}{\sin(n+1)\theta} \frac{1}{\operatorname{cotan} x\theta \operatorname{cotan}(x+1)\theta \ldots \operatorname{cotan}(x+n)\theta},$$

with many others of a similar nature.

SOME PROPERTIES OF TETRAHEDRA WHOSE OPPOSITE EDGES ARE AT RIGHT ANGLES TO ONE ANOTHER.

By *T. C. Lewis, M.A.*, Trinity College.

LET $ABCD$ be such a tetrahedron; A', B', C', D' the orthocentre of the faces. Let a, b, c be the sides of the triangle ABC, and d, e, f be the edges DA, DB, DC; δ_1, δ_2, δ_3, δ_4 the diameters of the circles circumscribing the four faces. We have evidently the lines DD', &c. perpendicular to the faces ABC, &c.

The first Proposition is one that was set as a Problem in the Trinity College Examination last June.

I. *The four spheres each circumscribing one face of the tetrahedron and having its centre at a distance from that face equal to half the distance of the fourth vertex are equal to one another.*

Since the circles about ABC, $D'BC$ are equal to one another, the sphere about ABC having its centre at the stated distance from ABC is equal to the sphere about $D'BC$, with centre at same distance ($\frac{1}{2}DD'$) from it; but this is the sphere about $DD'BC$, and the square on its diameter is $DD'^2 + \delta_4^2$.

Now (fig. 6)

$$3DD'^2 = d^2 + e^2 + f^2 - (D'A^2 + D'B^2 + D'C^2);$$

therefore $$3DD'^2 = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 - 3\delta_4^2,$$

since $$a^2 + D'A^2 = \delta_4^2,$$

therefore $$DD'^2 + \delta_4^2 = \tfrac{1}{3}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2).$$

Since this expression is symmetrical, the four spheres are equal to one another.

II. *The sum of the squares of two opposite edges is constant and equal to the square on the diameter of either of these spheres.*

For
$$\begin{aligned} BC^2 + AD^2 &= BC^2 + AD'^2 + DD'^2 \\ &= \delta_4^2 + DD'^2 = \text{constant} \\ &= \Delta^2. \end{aligned}$$

III. *The angular points A, B, C of any face, and the orthocentres A', B', C' of the opposite faces all lie on one sphere; and the four spheres so described are all equal.*

For the spheres about $ABCA'$ and $ABCB'$ are two spheres such as have been shown to be equal, and they have a common section ABC; they will therefore be coincident if the tangents from D are equal.

Now the tangent from D to the sphere $ABCA'$ is equal to the tangent from D to the circles about $A'BC$; if DA' produced meet BC in A'',

the square of this tangent $= 2DA'.DA''$
$$\begin{aligned} &= 2DB.DC\cos BDC \\ &= BD^2 + CD^2 - BC^3 \\ &= BD'^2 + CD'^2 + 2DD'^2 - BC^2, \\ &= 2DD'^2 - 2\delta_4^2 \cos A \cos B \cos C, \end{aligned}$$

where A, B, C are the angles of the face ABC; hence, the tangents from D to the three spheres $ABCA'$, $ABCB'$, $ABCC'$ are all equal, and these spheres are equal and have a common section; they are therefore one and the same sphere.

The tetrahedral coordinates of the centre of this sphere are $\frac{1}{2}p_1, \frac{1}{2}p_2, \frac{1}{2}p_3, -\frac{1}{2}p_4$; the fourth being deduced from the constant relation connecting the four coordinates of a point, and similar expressions hold for the centres of the other spheres; p_1, p_2, p_3, p_4 being the perpendiculars from the angular points of the tetrahedron upon the opposite faces.

IV. *The nine-point circles of the four faces lie on the same sphere.*

It is easily seen, that the sphere passing through the nine-point circle of one face and through the middle point of one of the other edges, will have as its section by either face

adjacent to that edge the nine-point circle to that face, whence the proposition easily follows.

Problem 6 in the Mathematical Tripos, Jan. 5, 1881, gives the same property in another form.

This might be called the twelve-point sphere of the tetrahedron.

V. *The radius of the twelve-point sphere is equal to a half of that of either of the four equal spheres already mentioned in Propositions I and III.*

For the triangle whose angular points are the middle points of DA, DB, DC is similar to ABC, and similar and equal to the triangle whose angular points are the middle points of BC, CA, AB, as well as parallel to it. The sections of the twelve-point sphere by these two planes are therefore parallel and equal; the centre of the twelve-point sphere is therefore half-way between these planes, the square on its diameters is therefore $\frac{1}{4}\delta_4^2 + \frac{1}{4}p_4^2$; whence the required result follows.

It may be remarked, that if the perpendiculars DD' from D upon the face ABC be indefinitely diminished, the properties of a triangle with reference to its orthocentre appear as particular cases of these more general theorems.

VI. *The four lines AA', BB', CC', DD' meet in a point.*

For AA' meets DD' in the orthocentre O of the triangle ADA'', and, as before,

$$\begin{aligned} 2DO.DD' &= DA''^2 + DA^2 - AA''^2 \\ &= DC^2 + DA^2 - AC^2 \\ &= DC^2 + DA^2 + DB^2 - \Delta^2, \text{ by II,} \end{aligned}$$

and this is symmetrical with respect to A, B and C; therefore DD' meets BB' and CC' also in the same point O.

ON THE HARMONIC SERIES AND STIRLING'S FORMULA.

By Professor *Paul Mansion.*

1. Consider an equilateral hyperbola $xy = 1$, and three points A, B, C on the curve (fig. 7) corresponding to the abscissæ $Oa = r$, $Ob = r + \frac{1}{2}$, $Oc = r + 1$. Draw the chord AC, the tangent $\alpha B\gamma$ terminating in α at the ordinate αA and

ı γ at the ordinate cC, and AD parallel to the axis of bscissæ meeting cC in D. Put

$$u_r = ABCD, \quad m_r = ABD, \quad M_r = A\alpha B\gamma D.$$

We find without difficulty

$$u_r = \frac{1}{r} - l\left(\frac{r+1}{r}\right),$$

or

$$\frac{1}{r} = l\left(\frac{r+1}{r}\right) + u_r;$$

whence

$$m_r = \tfrac{1}{2}\left(\frac{1}{r} - \frac{1}{r+1}\right),$$

$$M_r = \frac{1}{r} - \frac{1}{r+\frac{1}{2}} = m_r + \frac{1}{2r(r+1)(2r+1)}.$$

2. Consequently, we have also

$$\Sigma_n^\infty m_r = \tfrac{1}{2}\Sigma_n^\infty \left(\frac{1}{r} - \frac{1}{r+1}\right) = \frac{1}{2n} = \frac{1}{n} - \frac{1}{2n},$$

$$\Sigma_n^\infty M_r = \Sigma_n^\infty m_r + \Sigma_n^\infty \frac{1}{2r(r+1)(2r+1)} = \frac{1}{2n} + \Sigma_n^\infty \frac{1}{2r(r+1)(2r+1)}.$$

But

$$\Sigma_n^\infty \frac{1}{2r(r+1)(2r+1)} < \frac{1}{2(2n+1)} \Sigma \frac{1}{r(r+1)},$$

or

$$\Sigma_n^\infty \frac{1}{2r(r+1)(2r+1)} < \frac{1}{2n(2n+1)}.$$

Thus

$$\Sigma_n^\infty M_r < \frac{1}{2n} + \frac{1}{2n(2n+1)} \text{ or } \frac{1}{n} - \frac{1}{2n+1}.$$

3. We have, evidently,

$$m_r < u_r < M_r,$$

$$\Sigma_n^\infty m_r < \Sigma_n^\infty u_r < \Sigma_n^\infty M_r,$$

or

$$\frac{1}{n} - \frac{1}{2n} < \Sigma_n^\infty u_r < \frac{1}{n} - \frac{1}{2n+1}.$$

In particular, for $n = 1$, putting $C = \Sigma_1^\infty u_r$, we find

$$\tfrac{1}{2} < C < \tfrac{3}{4}.$$

4. Let $r = 1, 2, 3, \ldots, n-1$ in the relation

$$\frac{1}{r} = l\left(\frac{r+1}{r}\right) + u_r,$$

and add the results. We thus have

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n-1} = ln + \Sigma_1^{n-1} u_r = C + ln - \Sigma_n^{\infty} u_r.$$

Replacing $\Sigma_n^{\infty} u_r$ by the two values

$$\frac{1}{n} - \frac{1}{2n}, \quad \frac{1}{n} - \frac{1}{2n+1},$$

the one greater, the other smaller, we obtain

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} < C + ln + \frac{1}{2n} \quad \ldots\ldots\ldots\ldots\ldots(1),$$

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} > C + ln + \frac{1}{2n+1} \quad \ldots\ldots\ldots(2),$$

and, *à fortiori*,

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n} > C + ln + \frac{1}{2n+2} \quad \ldots\ldots\ldots(3).$$

The value of

$$H_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n}$$

is then given by the formulæ (1), (2) with an error less than

$$\frac{1}{2n} - \frac{1}{2n+1} = \frac{1}{4n^2 + 2n}.$$

5. Put $n = 1, 2, 3, \ldots N$ in (1), and add. We thus obtain successively

$$N + \tfrac{1}{2}(N-1) + \tfrac{1}{3}(N-2) + \ldots + \frac{1}{N}$$

$$< NC + l(1.2.3\ldots N) + \tfrac{1}{2}\left(1 + \tfrac{1}{2} + \ldots + \frac{1}{N}\right),$$

$$(N+1)\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{N}\right) - N$$

$$< NC + l(1.2.3\ldots N) + \tfrac{1}{2}\left(1 + \tfrac{1}{2} + \ldots + \frac{1}{N}\right),$$

$$(N + \tfrac{1}{2})\, H_N - N < nC + l(1.2.3\ldots N).$$

Replace in the first member H_N by the value

$$C + lN + \frac{1}{2N+1},$$

which is smaller, according to (2). We shall have, after some reductions,

$$l(1.2.3...N) > \tfrac{1}{2}(1+C) + (N+\tfrac{1}{2})\,lN - N.$$

We obtain also, by means of the inequalities (3) and (1),

$$l(1.2.3...N) < \tfrac{1}{2}(1+C) + (N+\tfrac{1}{2})\,lN - N + \tfrac{1}{2}.$$

Thus finally, θ being intermediate to 0 and 1,

$$l(1.2.3...N) = \tfrac{1}{2}(1+C) + (N+\tfrac{1}{2})\,lN - N + \tfrac{1}{2}\theta,$$

a very remarkable result, considering how elementary is the point of departure of No. 1.

ON A DIFFERENTIAL EQUATION.

By *H. J. Sharpe, M.A.*

In the Number of the *Messenger* for March last I discussed the solution of the Differential Equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y(x+A) = 0 \ldots\ldots\ldots\ldots\ldots(1).$$

I now propose to discuss the solution of

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y(x-A) = \frac{d}{dx}\,\frac{\cos(x+l)}{x+l} \ldots\ldots(2),$$

where A and l are constants, and A a certain function of l, which will presently be explained. I may remark that the latter equation like the former, arose in certain investigations in the Theory of Sound.

I shall shew that (2) can be solved in two distinct ways. The first solution will be in ascending series which converge for all values of x and A, but which for numerical calculation are only fitted for small values of x and A.

The second solution will be in semi-converging series of descending powers of x and A, and which therefore are very suitable for large values of x, the case to me of greatest interest and use.

I will first explain what function A is of l.

Let y_1 be that solution of

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+(x-A)y=0 \quad \ldots\ldots\ldots\ldots(3).$$

which does not involve $\log x$.

Then it will be seen by my former paper that we may take

$$y_1=\frac{2}{\pi}\left(\varepsilon^{\frac{1}{2}(\pi A)}+\varepsilon^{-\frac{1}{2}(\pi A)}\right).\int_0^\infty \cos\left(x\frac{d}{s}-A\phi\right).\frac{d\phi}{s} \ldots(4),$$

where for brevity d has been put for $\varepsilon^{\phi}-\varepsilon^{-\phi}$ and s for $\varepsilon^{\phi}+\varepsilon^{-\phi}$. y_1 is such that it $=1$ when $x=0$. This has been proved, and A in (2) is supposed to be such that

$$\frac{dy_1}{dx}=0, \text{ when } x=-l \ldots\ldots\ldots\ldots (5),$$

y_1 having the value (4). I will now shew that the solution of (2) can be found in the following form

$$y=Cy_1\log(x+l)+z \ldots\ldots\ldots\ldots(6),$$

where C is a constant to be determined and z can be found in a converging series of powers of x. Substituting from (6) into (2) and putting for shortness D for

$$\left(x\frac{d^2}{dx^2}+\frac{d}{dx}+x\right),$$

(2) becomes

$$(D-A)z+2C\frac{dy_1}{dx}-\frac{2Cl\frac{dy_1}{dx}}{x+l}+\frac{Cly_1}{(x+l)^2}=\frac{d}{dx}\frac{\cos(x+l)}{x+l} \ldots(7).$$

From (4) we may get

$$y_1=\phi(x)=\phi(-l+x+l)=L_0+L_1(x+l)+L_2(x+l)^2+\&c.,$$

which may readily be shown to be a converging series,

$$L_n=\frac{1}{\lfloor n}\frac{d^ny_1}{dx^n} \text{ when } x=-l.$$

From (5) it is important to notice that $L_1=0$.

Then in (7) it will be found that

$$-\frac{2Cl\frac{dy_1}{dx}}{x+l}+\frac{Cly_1}{(x+l)^2}=-2Cl[2L_2+3L_3(x+l)+4L_4(x+l)^2+\&c.]$$
$$+Cl\left[\frac{L_0}{(x+l)^2}+L_2+L_3(x+l)+\&c.\right].$$

Now the right-hand of (2) when expanded is

$$= \frac{-1}{(x+l)^2} - \frac{1}{\lfloor 2} + \frac{3}{\lfloor 4}(x+l)^2 - \frac{5}{\lfloor 6}(x+l)^4 + \&c.$$

Let $$ClL_0 = -1,$$

hen it will be found that with this value of C the term nvolving $\frac{1}{(x+l)^2}$ disappears from (7) and that equation becomes

$$(D-A)z = -2C\frac{dy_1}{dx} + 2Cl[2L_2 + 3L_3(x+l) + 4L_4(x+l)^2 + \&c.]$$

$$- Cl[L_2 + L_3(x+l) + \&c.] \ldots\ldots\ldots\ldots (8).$$

From (4) obviously $\frac{dy_1}{dx}$ can be expanded in a converging series of powers of x, whatever be the value of x, and the same is true of the series involving the L's, although this latter point requires, I think, some consideration. It might be imagined, from the way these series arise, that they could not be expressed in converging series of powers of x unless x were $< l$, but this restriction need not be imposed. It will be found on examination that the case is exactly parallel to this one.

The functions $\frac{\sin(x+l)}{x+l}$ and $\frac{1-\cos(x+l)}{(x+l)^2}$ can be expanded in converging series of powers of x, whatever be the value of x. This can be proved thus. Expand the first function in ascending powers of $(x+l)$. Expand the powers of $(x+l)$. It will be found that by sufficiently increasing n, the ratio of the coefficient of x^{n+1} to that of x^n can be made less than any assignable quantity. It will be found that exactly similar reasoning can be applied to the series in (8). We may write, therefore, (8) in the following form,

$$(D-A)z = a_0 + a_1x + a_2x^2 + \&c.,$$

a converging series. z can be found from this in an ascending series of powers of x, in a series which is ultimately more convergent than the series on the right-hand. Thus (6) is established, and is true for all values of x and A. As the series for z ascends in powers of x and A, it is unsuitable for numerical calculation when x and A are at all large. We proceed, therefore, to get a descending series for y in (2).

In (2) put

$$y = \varepsilon^{ix} v + \varepsilon^{-ix} u,$$

where v and u are two new functions of x and $i = \sqrt{(-1)}$.

Express the right-hand side of (2) in terms of ε^{ix} and ε^{-ix}. Equate the coefficients of ε^{ix} and ε^{-ix} respectively on each side, and we get

$$x\frac{d^2v}{dx^2} + \frac{dv}{dx}(2ix+1) + (i-A)v = \frac{ie^{il}}{2(x+l)} - \frac{\varepsilon^{il}}{2(x+l)^2} \ldots (9),$$

$$x\frac{d^2u}{dx^2} + \frac{du}{dx}(-2ix+1) - (i+A)u = -\frac{ie^{-il}}{2(x+l)} - \frac{e^{-il}}{2(x+l)^2}.$$

Suppose x very large compared with l, then the right-hand side of (9) can be expanded in a converging series of powers of $\frac{l}{x}$. Let the series be

$$a_1 x^{-1} + a_2 x^{-2} + \&c.$$

Assume $$v = \alpha_1 x^{-1} + \alpha_2 x^{-2} + \&c. \ldots\ldots\ldots\ldots\ldots (10).$$

It will be found that we get the following equations for determining α_1, α_2, &c.

$$-(i+A)\alpha_1 = a_1,$$

$$-(3i+A)\alpha_2 + \alpha_1 = a_2,$$

$$\&c. \qquad \&c.$$

$$-\{(2n-1)i+A\}\alpha_n + (n-1)^2\alpha_{n-1} = a_n.$$

Thus the series (10), though ultimately divergent, begins by converging rapidly when x and A are at all large, and we know that the converging part of such series may be safely used for the numerical calculation of v.

We may also notice another remarkable distinction between the two solutions.

The first is true for *all* values of x provided that A has the value derived from (5).

The second is true only for values of x large compared with l, but for *any* value of A, for in the latter solution no use was made of condition (5).

Whether a solution of (2) could be obtained in a calculable form true for *all* values of x, whatever value A have, I will not undertake to say. The condition (5) occurred in my sound investigations.

FORMULÆ FOR THE sn, cn, dn OF $u+v+w$.

By *J. W. L. Glaisher.*

§ 1. As the formulæ for $\operatorname{sn}(u+v+w)$, $\operatorname{cn}(u+v+w)$, $\operatorname{dn}(u+v+w)$ as symmetrical functions of the sn's, cn's, dn's of u, v, w may be sometimes required, and, so far as I know, have not been hitherto published, it seems worth while to place them on record here.

Writing s, c, d for sn, cn, dn and using the suffixes 1, 2, 3 to refer to the arguments u, v, w, so that s_1 denotes $\operatorname{sn} u$, &c., the formulæ are

$$\operatorname{sn}(u+v+w)=\frac{A}{D},$$

$$\operatorname{cn}(u+v+w)=\frac{B}{D},$$

$$\operatorname{dn}(u+v+w)=\frac{C}{D},$$

where

$$\begin{aligned}
A= \;& s_1s_2s_3\{-1-k^2+2k^2(s_1^2+s_2^2+s_3^2)\\
&\quad-(k^2+k^4)(s_2^2s_3^2+s_3^2s_1^2+s_1^2s_2^2)+2k^4s_1^2s_2^2s_3^2\}\\
&+ s_1c_2c_3d_2d_3(\;1+k^2s_2^2s_3^2-k^2s_3^2s_1^2-k^2s_1^2s_2^2)\\
&+ s_2c_3c_1d_3d_1(\;1-k^2s_2^2s_3^2+k^2s_3^2s_1^2-k^2s_1^2s_2^2)\\
&+ s_3c_1c_2d_1d_2(\;1-k^2s_2^2s_3^2-k^2s_3^2s_1^2+k^2s_1^2s_2^2),\\
B= \;& c_1c_2c_3\quad(\;1-k^2s_2^2s_3^2-k^2s_3^2s_1^2-k^2s_1^2s_2^2+2k^4s_1^2s_2^2s_3^2)\\
&+ s_2s_3d_2d\,c_1(-1+k^2s_2^2s_3^2-k^2s_3^2s_1^2-k^2s_1^2s_2^2+2k^2s_1^2)\\
&+ s_3s_1d_3d_1c_2(-1-k^2s_2^2s_3^2+k^2s_3^2s_1^2-k^2s_1^2s_2^2+2k^2s_2^2)\\
&+ s_1s_2d_1d_2c_3(-1-k^2s_2^2s_3^2-k^2s_3^2s_1^2+k^2s_1^2s_2^2+2k^2s_3^2),\\
C= \;& d_1d_2d_3\quad(\;1-k^2s_2^2s_3^2-k^2s_3^2s_1^2-k^2s_1^2s_2^2+2k^2s_1^2s_2^2s_3^2)\\
&+ k^2s_2s_3c_2c_3d_1(-1+k^2s_2^2s_3^2-k^2s_3^2s_1^2-k^2s_1^2s_2^2+2s_1^2)\\
&+ k^2s_3s_1c_3c_1d_2(-1-k^2s_2^2s_3^2+k^2s_3^2s_1^2-k^2s_1^2s_2^2+2s_2^2)\\
&+ k^2s_1s_2c_1c_2d_3(-1-k^2s_2^2s_3^2-k^2s_3^2s_1^2+k^2s_1^2s_2^2+2s_3^2),\\
D= \;& 1-2k^2(s_2^2s_3^2+s_3^2s_1^2+s_1^2s_2^2)\\
&+4(k^2+k^4)s_1^2s_2^2s_3^2-2k^4s_1^2s_2^2s_3^2(s_1^2+s_2^2+s_3^2)\\
&+k^4(s_2^4s_3^4+s_3^4s_1^4+s_1^4s_2^4).
\end{aligned}$$

§ 2. The formulæ were calculated as follows:
We have

$$\operatorname{sn}(u+v+w) = \frac{\operatorname{sn}(u+v)\, c_3 d_3 + s_3 \operatorname{cn}(u+v) \operatorname{dn}(u+v)}{1 - k^2 s_3^{\,2} \operatorname{sn}^2(u+v)}.$$

To render this expression symmetrical with respect to u, v, w, multiply the numerator and denominator by

$$1 - k^2 s_3^{\,2} \operatorname{sn}^2(u-v);$$

the denominator thus becomes

$$1 - k^2 s_3^{\,2} \{\operatorname{sn}^2(u+v) + \operatorname{sn}^2(u-v)\} + k^4 s_3^{\,4} \operatorname{sn}^2(u+v) \operatorname{sn}^2(u-v),$$

which, when multiplied by $(1 - k^2 s_1^{\,2} s_2^{\,2})^2$,

$$= (1 - k^2 s_1^{\,2} s_2^{\,2})^2 - 2k^2 s_3^{\,2} (s_1^{\,2} c_2^{\,2} d_2^{\,2} + s_2^{\,2} c_1^{\,2} d_1^{\,2}) + k^4 s_3^{\,4} (s_1^{\,2} - s_2^{\,2})^2,$$

and this expression may be reduced to the form D.

The numerator, when multiplied by $1 - k^2 s_3^{\,2} \operatorname{sn}^2(u-v)$,

$$\begin{aligned} &= c_3 d \operatorname{sn}(u+v) + s_3 \operatorname{cn}(u+v) \operatorname{dn}(u+v) \\ &\quad - k^2 s_3^{\,2} c_3 d_3 \operatorname{sn}(u+v) \operatorname{sn}(u-v) \operatorname{sn}(u-v) \\ &\quad - k^2 s_3^{\,3} \operatorname{sn}(u-v) \operatorname{cn}(u+v) \operatorname{sn}(u-v) \operatorname{dn}(u+v), \end{aligned}$$

which, when multiplied by $(1 - k^2 s_1^{\,2} s_2^{\,2})$,

$$\begin{aligned} &= c_3 d_3 (s_1 c_2 d_2 + s_2 c_1 d_1)(1 - k^2 s_1^{\,2} s_2^{\,2}) + s_3 (c_1 c_2 - s_1 s_2 d_1 d_2)(d_1 d_2 - k^2 s_1 s_2 c_1 c_2) \\ &- k^2 s_3^{\,2} c_3 d_3 (s_1^{\,2} - s_2^{\,2})(s_1 c_2 d_2 - s_2 c_1 d_1) - k^2 s_3^{\,3} (s_1 c_1 d_2 - s_2 c_2 d_1)(s_1 d_1 c_2 - s_2 d_2 c_1); \end{aligned}$$

and this expression may be reduced to the form A.

The expressions B and C are obtained in a similar manner by multiplication by the same quantity, viz.

$$(1 - k^2 s_1^{\,2} s_2^{\,2})^2 \{1 - k^2 s_3^{\,2} \operatorname{sn}^2(u-v)\},$$

and reduction.

§ 3. If u, v, w, x be any four quantities whose sum is zero, and if the suffix 4 refer to the argument w, then it is a known* theorem that

$$k'^2 - k^2 k'^2 s_1 s_2 s_3 s_4 + k^2 c_1 c_2 c_3 c_4 - d_1 d_2 d_3 d_4 = 0,$$

and therefore

$$\begin{aligned} &k'^2 + k^2 k'^2 s_1 s_2 s_3 \operatorname{sn}(u+v+w) \\ &\qquad + k^2 c_1 c_2 c_3 \operatorname{cn}(u+v+w) - d_1 d_2 d_3 \operatorname{dn}(u+v+w) = 0, \end{aligned}$$

* Gudermann, *Crelle's Journal*, vol xviii. p. 164, and Cayley, *Proceedings of the London Mathematical Society*, vol x. pp. 43 48. See also *Messenger*, vol x. p. 130.

that is

$$k'^2(D+k^2s_1s_2s_3A)=d_1d_2d_3C-k^2c_1c_2c_3B.$$

As a verification of the foregoing values of A, B, C, D they were substituted in this formulæ, and found to satisfy it, each of the two members of the equation being equal to $k'^2\times$

$$k^2s_1s_2s_3\Sigma s_1c_2c_3d_2d_3(1+k^2s_2^2s_3^2-k^2s_3^2s_1^2-k^2s_1^2s_2^2)$$
$$+1-2k^2\Sigma s_1^2s_2^2+(3k^2+3k^4)\,s_1^2s_2^2s_3^2+k^4\Sigma s_1^4s_2^4$$
$$-(k^4+k^6)\,s_1^2s_2^2s_3^2\Sigma s_1^2s_2^2+2k^6s_1^4s_2^4s_3^4.$$

§ 4. If we put $u=v=w$, the expressions of A, B, C, D reduce respectively to

$$s\{3-(4+4k^2)\,s^2+6k^2s^4-k^4s^8\},$$
$$c\{1-4s^2+6k^2s^4-4k^4s^6+k^4s^8\},$$
$$d\{1-4k^2s^2+6k^2s^4-4k^2s^6+k^4s^8\},$$
$$1-6k^2s^4+(4k^2+4k^4)\,s^6-3k^4s^8,$$

which are their known values in this case (Cayley's *Elliptic Functions*, p. 77).

§ 5. Since the preceding portion of this note was in type I have found that in *Crelle's Journal* for 1851* Cayley has given a symmetrical formulæ for the sn, cn, dn of $u+v+w$. Denoting the three numerators by A', B', C', and their common denominator by D', Cayley's formulæ, expressed in the notation of the present note, are

$$A'=s_1c_2c_3d_2d_3+s_2c_3c_1d_3d_1+s_3c_1c_2d_1d_2$$
$$-s_1s_2s_3(1+k^2-k^2s_1^2-k^2s_2^2-k^2s_3^2+k^4s_1^2s_2^2s_3^2),$$

$$B'=c_1c_2c_3\,(1-k^4s_1^2s_2^2s_3^2)-d_1d_2d_3(s_2s_3c_1d_1+s_3s_1c_2d_2+s_1s_2c_3d_3),$$

$$C'=d_1d_2d_3(1-k^2s_1^2s_2^2s_3^2)-k^2c_1c_2c_3\,(s_2s_3c_1d_1+s_3s_1c_2d_2+s_1s_2c_3d_3),$$

$$D'=1-k^2(s_2^2s_3^2+s_3^2s_1^2+s_1^2s_2^2)+(k^2+k^4)\,s_1^2s_2^2s_3^2$$
$$-k^2s_1s_2s_3(s_1c_2c_3d_2d_3+s_2c_3c_1d_3d_1+s_3c_1c_2d_1d_2).$$

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* "Note sur l'addition des fonctions elliptiques," vol. XLI. pp. 57–65. The paper also contains an expression for sn $(u+v+w+x)$.

These expressions are very much shorter than those given in § 1, and the forms of B' and C' are very elegant. In the formulæ of § 1, however, the denominator is a rational function of s_1^2, s_2^2, s_3^2, so that in spite of their greater length these may be regarded as forming the standard system.

It follows from § 3 that A', B', C', D' must be connected by the equation

$$k'^2(D' + k^2 s_1 s_2 s_3 A') = d_1 d_2 d_3 C' - k^2 c_1 c_2 c_3 B',$$

and, on putting for A', B', C', D' their values, we find that each member of this equation

$$= k'^2 (1 - k^2 s_2^2 s_3^2)(1 - k^2 s_3^2 s_1^2)(1 - k^2 s_1^2 s_2^2).$$

If we put $u = v = w$, we find that A', B', C', D' reduce directly to the four expressions written in § 4; and it may also be observed that if we put $w = 0$, the formulæ reduce to the standard form (*i.e.* in which the denominator is $1 - k^2 s_1^2 s_2^2$), and not to one of the forms in which the denominator involves c's and d's.*

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, June 9th, Samuel Roberts, Esq., F.R.S., *President*, in the Chair. Prof. Mannheim (Foreign Member) and Mr. T. Craig, of the United States Coast Survey, were admitted into the Society, and Mr. G. R. Dick, M.A., late Fellow of Caius College, Cambridge, Professor of Mathematics in the Royal College, Mauritius, was elected a Member. Prof. Mannheim communicated a paper "Sur les surfaces parallèles." Dr. Hirst, F.R.S. proposed a vote of thanks to the author which was unanimously accorded. M. Mannheim briefly thanked the meeting. Other communications were: "On certain symbolic operators," Mr. J. W. L. Glaisher, F.R.S.; "On a system of coordinates," Prof. Genese; "Note on a system of Cartesian ovals passing through four points on a circle," Mr. R. A. Roberts; "On the Gaussian theory of surfaces," Prof. Cayley, F.R.S.; "On a theorem in the calculus of operations," Mr. J. J. Walker; "On spherical quartics, with a quadruple cyclic arc and a triple focus," Mr. H. M. Jeffery, F.R.S.; "Note on the wave surface," Prof. Mannheim.

R. Tucker, M.A., *Hon. Sec.*

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* See *Messenger*, vol. x. p. 106 (November, 1880).

TRANSFORMATION OF A BIORDINAL OF SCHWARZ'S.

By *Sir James Cockle, M.A., F.R.S.*

1. THE biordinal of Schwarz's, here to be transformed, is that discussed by Professor Cayley (*Q. J. of Math.* vol. XVI. p. 268), in whose paper the first $-$ should be $+$. The misprint of course does not otherwise affect the paper.

2. The biordinal in question may be written

$$x(1-x)\frac{d^2y}{dx^2}+(\tfrac{2}{3}-\tfrac{7}{6}x)\frac{dy}{dx}+\tfrac{1}{48}y=0.$$

3. If we change x into $-t^2$ and then replace t by x, Schwarz's biordinal becomes

$$(1+x^2)x^2\frac{d^2y}{dx^2}+\tfrac{1}{3}(1+4x^2)x\frac{dy}{dx}-\tfrac{1}{12}x^2y=0\ldots\ldots(1).$$

4. If in the equation (11) of Mr. Rawson's paper (*ante*, p. 21) we put $a=\frac{1}{2}$ and $x_1=\dfrac{x}{3\sqrt{3}}$, it becomes

$$\frac{d^2u}{dx^2}+\frac{\frac{5}{3}x}{1+x^2}\frac{du}{dx}+\tfrac{1}{12}\frac{1}{(1+x^2)^2}u=0.$$

5. Among the (infinite number of) ways in which this last equation can be put under the binomial form two, obtained by means of an artifice of Boole (*Diff. Eq.*, 1865, pp. 432, 433), are conspicuous.

6. Put $u=(1+x^2)^m z$, then the equation in u takes the form

$$(1+x^2)x^2\frac{d^2z}{dx^2}+(4m+\tfrac{5}{3})x^3\frac{dz}{dx}+\left[2m+\frac{\{4m(m-1)+\frac{10}{3}m\}x^2+\frac{1}{12}}{1+x^2}\right]x^2z=0.$$

7. First, let $m=\frac{1}{4}$; we get

$$(1+x^2)x^2\frac{d^2z}{dx^2}+\tfrac{8}{3}x^3\frac{dz}{dx}+\tfrac{7}{12}x^2z=0\ldots\ldots\ldots(2).$$

8. Secondly, let $m = -\frac{1}{12}$; we get

$$(1+x^3)\,x^2\frac{d^2z}{dx^2} + \tfrac{4}{3}x^3\frac{dz}{dx} - \tfrac{1}{12}x^2z = 0 \ldots\ldots\ldots (3).$$

9. A symbolical form of (1) is

$$D(D-\tfrac{2}{3})\,y + x^2(D+\tfrac{1}{2})(D-\tfrac{1}{6})\,y = 0 \ldots\ldots (\mathrm{I}),$$

wherein I leave the x^2 on the left.

10. So, a symbolical form of (2) is

$$D(D-1)\,z + x^2(D+\tfrac{1}{2})(D+\tfrac{7}{6})\,z = 0 \ldots\ldots (\mathrm{II}).$$

11. And a symbolical form of (3) is

$$D(D-1)\,z + x^2(D+\tfrac{1}{2})(D-\tfrac{1}{6})\,z = 0 \ldots\ldots (\mathrm{III}).$$

12. Now I say that (II), or (III), is a transformation of (I).

13. Consider the binomial biordinal

$$(D-\beta_1)(D-\beta_2)\,u + (D-\alpha_1)(D-\alpha_2)\,x^2u = 0.$$

14. I have shown (*Proc. L. M. Soc.* vol. XIII. pp. 69, 70) that we can change β_2 into $\alpha_1+\alpha_2-\beta_2$ without affecting the α's or β_1. And in precisely the same way it can be shown that β_1 may be changed into $\alpha_1+\alpha_2-\beta_1$. So that either of the β's, or both simultaneously, may be changed from β into $\alpha_1+\alpha_2-\beta$.

15. The same is true of

$$(D-\beta_1)(D-\beta_2)\,u + x^2(D-\alpha_1)(D-\alpha_2)\,u = 0,$$

which is a transformation of the preceding biordinal. This last form is sometimes the more convenient, and a reference to it is to be implied in what follows.

16. In (I) we have $\beta_2 = \frac{2}{3}$, $\alpha_1 = -\frac{1}{2}$, $\alpha_2 = \frac{1}{6}$ and

$$\alpha_1+\alpha_2-\beta_2 = -\tfrac{1}{3}-\tfrac{2}{3} = -1.$$

Consequently we can transform (I) into

$$D(D+1)\,v + x^2(D+\tfrac{1}{2})(D-\tfrac{1}{6})\,v = 0.$$

17. Put $v=(D+1)^{-1}z$, and we get (III), which is thus seen to be a transformation of Schwarz's biordinal. Mr. Rawson's paper constitutes an addition to the theory of differential resolvents.

18. If we denote the z's of (2) and (3) by z_2 and z_3 respectively, we have $(1+x^2)^{\frac{1}{2}}z_2=(1+x^2)^{-1\frac{1}{2}}z_3$. We can also pass from (II) to (III), or *vice versa*, by the transformation of Art. 14.

19. The change of x into x^{-1} converts the biordinal of Art. 15 into

$$(D+\alpha_1)(D+\alpha_2)u+x^2(D+\beta_1)(D+\beta_2)u=0,$$

whence (see Art. 14) we can pass to

$$(D+\alpha_1)(D+\alpha_2)\dot{u}+x^2(D+\alpha_1+\alpha_2-\beta_1)(D+\alpha_1+\alpha_2-\beta_2)\dot{u}=0,$$

and thence, by changing D into $D-\alpha_1-\alpha_2$, to

$$(D-\alpha_1)(D-\alpha_2)v+x^2(D-\beta_1)(D-\beta_2)v=0.$$

20. I have thought it well to bring these transformations into view, because, although Boole has used a form of one of them in dealing with the equation of Laplace's Functions, yet he (*vid. op. cit.* p. 435, line 7 from bottom) notices it only as a particular artifice, not as part of a general system of transformation.

21. It follows from what has been said (in Arts. 14 and 19) that $-\alpha$ can be changed into $-\beta_1-\beta_2+\alpha$, or, which is the same thing, α can be changed into $\beta_1+\beta_2-\alpha$; and this applies to the change of each α, or of both simultaneously.

22. The simplest communication between (I) and (II) seems to lie through (III).

23. The term first resolvent, used by Mr. Rawson, is expressive and better than the word subresolvent, used by me (vol. III. p. 109).

24. It is not worth while to proceed further with the two equations upon the discussion of which I entered at pp. 248–9 of vol. III. (old series). For the more general equation

$$\frac{d^2z}{dt^2}+\frac{a}{t^2\{\log(Ct^m)\}^n}z=0$$

can, when $n=1$ or 2, be reduced to the binomial form. Let $t\frac{d}{dt}=D$, and we have

$$D(D-1)z+\frac{a}{\{\log(Ct^m)\}^n}z=0,$$

which the change of Ct^m into x (i.e. of t into $C^{-\frac{1}{m}}x^{\frac{1}{m}}$) converts into

$$mD(mD-1)z+\frac{a}{(\log x)^n}z=0,$$

wherein $D=x\frac{d}{dx}$. Let $x=e^\theta$, and we get

$$m\frac{d}{d\theta}\left(m\frac{d}{d\theta}-1\right)z+\frac{a}{\theta^n}z=0,$$

which is binomial when $n=1$ or 2.

2, Sandringham Gardens,
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ERRATA.

New Series, vol. v. p. 26, art. 4, line 7: the first $\sqrt{\infty}$ should be ∞.
Vol. x. p. 98, art. 2, line 5: the last two terms of the numerator of the last fraction should read

$$-\frac{n^2}{a}v^{n-1}+(n-1)^2xv^{n-2}.$$

p. 99, art. 3, line 7, *for* "$(\varepsilon-\zeta-1,)$" *read* $(\varepsilon-\zeta)-1$.
p. 100, from last line of art. 8 to end of paper, *for* U, U_2 *read* U^{-1}, U_2^{-1}.

ON THE DISTANCE OF RAINBOWS.

By *Homersham Cox, M.A.*, Jesus College, Cambridge.

IN favorable circumstances the distance of the raindrops by which rainbows are formed may be ascertained with considerable precision. Suppose, for instance, the observer sees the slope of a hill tinted with the colours of the bow, while a nearer object, such as the roof of a house, abruptly intercepts it. In such a case it is manifest that the bow is formed somewhere between the nearer and the remoter objects.

The effect of the linear distance of the raindrops on the brightness of the bow does not seem to have been noticed in works on Optics.

There are two accepted theories of the rainbow. The first is due to Descartes, and was completed by Newton so far as relates to the order of the colours (Verdet, *Leçons*

D'Optique Physique, § 104. *English Cyclopædia Art. Rainbow*). The essential assumption in this theory is that the rays which produce a sensible effect on the eye are those which emerge from the raindrops in a state of parallelism. Other rays which emerge in a state of divergence are assumed by Descartes to produce an effect which is inconsiderable.

The more exact and complete theory of Professor Airy is founded on the principles of the Undulatory Theory. The investigations of Professor Airy first appeared in the *Cambridge Philosophical Transactions*, and a remarkably clear account of them is given by Verdet in his treatise already cited. For the very simple purpose of this paper the Cartesian theory will be sufficiently accurate.

The following remarks relate only to the primary rainbow. For the sake of simplicity let us consider only the highest part of the rainbow: the conclusions will be precisely the same for other parts. Also, for the present, let the attention be confined to light of one colour only—say, red.

In fig. 8 suppose E to be the eye of the observer; RA the direction of a ray of red light proceeding from the lowest point of the sun and deflected to E by a raindrop at A. Let RA produced meet a horizontal plane through E at H. Let HS be the direction of a red ray proceeding from the highest point of the sun; and let that ray be deflected to E by a raindrop at B. The angle RHS is the sun's angular breadth or about 32′.

It is shown in treatises on Optics that where rays emerge from the raindrop in a state of parallelism, the angle through which rays of any particular colour are deflected depends solely on the index of refraction. Hence the angles at B are equal to the angles at A.

It follows geometrically that the angle AEB is equal to the angle RHS. In other words, if we adopt the Cartesian theory as sufficiently correct, the angular breadth of any one colour of the bow is equal to the sun's angular breadth.

All the raindrops producing the red colour are contained between two conical surfaces, of which AEH and BEH are the semi-vertical angles.

In fig. 9 let E be the eye; Eb a horizontal line; $FabG$ a vertical section of a thin stratum of raindrops; the angle GEb being the same as the angle AEB in the last figure. Fa and Gb are supposed to be arcs of circles having their common centre at E. These arcs are sections of spheres having the same centre.

The whole stratum producing the entire red bow bounded by the earth's surface is comprised between the two spheres, the two conical surfaces, and the horizontal plane through *Eb*.

If *ab* is comparatively small, the solid content of this stratum, *cæteris paribus*, is proportional to the square of the radius *Ea*.

Also, since the angle at which the red rays are deflected is constant, the quantity of light so deflected is proportional to the solid content of the stratum, and therefore to the square of the distance from the eye.

But whatever the distance of the stratum, the size of the image on the retina is the same. It follows that the intrinsic brightness of the image produced by a stratum of raindrops of a given thickness is proportional to the quantity of light concentrated on the eye, and therefore to the square of the distance of the stratum.

This result will be considerably modified by the absorption of light passing through the imperfectly transparent air. But, notwithstanding that absorption, rainbows formed by distant raineclouds are generally far more brilliant than those which are formed comparatively near the eye of the observer. Rainbows formed artificially by fountains are, for the same reasons, less vivid than those formed by rainclouds.

GEOMETRICAL NOTES.

By *A. F. Torry, M.A.*

It is a well known theorem that "the quadrilateral formed by the tangents to a conic from its foci has for its other vertices the imaginary foci on the minor axis and the circular points at infinity."

It it interesting to observe how, by representing this imaginary construction by a real figure, other theorems are readily deducible.

Of the vertices of a quadrilateral circumscribing an ellipse let F, F' (fig. 10) represent the real foci, S, S' the imaginary foci, and M, N the circular points at infinity. Let A, B, C, D be the points of contact of FSN, MFS', $S'F'N$, $F'SM$ respectively. The centre O will be the intersection of SS', FF' and hence also of AC, BD. The directrices will be represented by AB, CD.

(1) Consider the straight lines FA, FB the tangents from F. Together they constitute a conic, not an hyperbola, because the lines are imaginary; but a circle, because it passes through M, N the circular points at infinity; and having double contact with the ellipse at A, B, that is, at the points of their intersection with the directrix corresponding to F.

We deduce therefore that an indefinitely small circle at the focus has double contact with the conic at the imaginary points where the corresponding directrix intersects them both.

(2) How shall we represent the *director circle?* It is the locus of tangents to the ellipse which intersect at right angles. Such a pair of tangents together with the straight lines joining their point of intersection to M and N will therefore form a harmonic pencil.

Hence if P, Q be two points on MN making the range $[MPNQ]$ harmonic, the locus of the intersection of tangents from P and Q will be the required curve—the director circle.

Where will it cut the given ellipse? At the points of intersection tangents to the ellipse must coincide as well as be at right angles. This is a property peculiar to straight lines through the circular points. Hence A, B, C, D are the points sought for: the points which we have already found to be the intersections of the ellipse by its directrices.

The curve being a circle will pass through M, N and have O for the pole of MN.

In ordinary Geometry "a directrix of a conic is the radical axis of the indefinitely small circle at the corresponding focus and the director circle." Because in our figure all pass through the points AB.

Mr. Genese has recently called attention to this property.

(3) The properties we have just discussed of the curve representing the director circle may be more generally stated as follows:

If a pair of tangents be drawn to a conic, so that when produced to meet a given straight line they make with two fixed points on that line a harmonic range, they will intersect on a conic passing through the two fixed points and intersecting the given conic where the tangents from the fixed points meet it.

(4) Many other theorems may be readily deduced. It will suffice to state one of them, obtained by making M and N real points at infinity.

Tangents drawn to a conic (A) parallel to conjugate diameters of a second conic (B) will intersect on a third conic (C), concentric with A, and similar and similarly situated to (B). Also (C) and (A) will intersect where the tangents to (A) are parallel to the asymptotes of (B).

The indefinitely small circle at F, $(x-ae)^2+y^2=0$, is equivalent to $x-ae=\pm y\sqrt{(-1)}$, the two straight lines FA, FB. Its equation may also be written

$$(1-e^2)(a^2-x^2)-y^2=(ex-a)^2.$$

It therefore has double contact with $y^2=(1-e^2)(a^2-x^2)$, where the directrix $ex=a$ intersects them both. And, lastly, the coordinates of the points A, B, C, D, viz. $\pm\frac{a}{e}$, $\pm\frac{b^2}{ae}\sqrt{(-1)}$, satisfy all the above equations and that of the circle

$$x^2+y^2=a^2+b^2.$$

ON A TRANSCENDENTAL DIFFERENTIAL EQUATION.

By *H. J. Sharpe, M.A.*

1. Supposing we have the following Differential Equation of an infinite order in x,

$$\frac{2}{\pi}\{\varepsilon^{\frac{1}{2}\pi D}+\varepsilon^{-\frac{1}{2}\pi D}\}\int_0^\infty \cos\left(l\frac{d}{s}+D\phi\right).\frac{d\phi}{s}.Y=\frac{\cos(x+l)}{x+l}\ldots(1),$$

where D is a symbol of operation denoting $\left(x\frac{d^2}{dx^2}+\frac{d}{dx}+x\right)$, d stands for $\varepsilon^{\phi}-\varepsilon^{-\phi}$, s for $\varepsilon^{\phi}+\varepsilon^{-\phi}$, and l is independent of x. It is required to find Y as a function of x. I have already proved that the operating symbol, the coefficient of Y, can be expanded in a series of ascending powers of l and D. The complete expansion, together with the laws of its formation, is given in (22) p. 180 of No. 119 of last March, putting there D for A. On p. 183 I have further proved that if (supposing, for a moment, D to be a symbol of quantity) we equate this expansion to zero, then, considered as an equation in D, it has all its roots real. Consequently we may consider the coefficient of Y as the product of an infinite number of factors

$$(D-A_1)(D-A_2)\ldots,$$

where A_1, A_2 are all real and certain functions of l. (1) therefore may be solved by separation of symbols. I shall prove the following remarkable property of this equation. First, if $x<l$, Y is a certain very complicated function of x and l. Secondly, if $x>l$, $Y=\frac{\cos x}{x}$, which we observe is independent of l. The latter, as being the simpler case, I shall take first.

2. Suppose then $x>l$. Let the coefficient of Y in (1), expanded in ascending powers of l, be

$$1+a_1l+a_2l^2+\&c.=z \text{ suppose,}$$

where a_1, a_2, &c. are functions of D, regarded for a moment as a symbol of quantity. It is easily shown that a_1, a_2, &c. satisfy the following equations:

$$\left.\begin{aligned} a_1 &= -D \\ 2^2a_2+1 &= -Da_1 \\ 3^2a_3+a_1 &= -Da_2 \\ 4^2a_4+a_2 &= -Da_3 \\ \&c. &= \&c. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (2).$$

For z satisfies the equation

$$l\frac{d^2z}{dl^2}+\frac{dz}{dl}+lz+Dz=0 \quad \ldots\ldots\ldots\ldots\ldots (3).$$

This equation is the same as (1) p. 175.

Similarly, expanding the right-hand side of (1) in ascending powers of l by Taylor's theorem, (1) becomes

$$(1+a_1l+a_2l^2+\&c.)\,Y=\left(1+l\frac{d}{dx}+\frac{l^2}{2!}\frac{d^2}{dx^2}+\&c.\right)\frac{\cos x}{x} \ldots (4).$$

The right-hand side is convergent, because $x>l$. On the left, a_1, a_2, &c. are rational integral functions of D. Their values can be got from (22) p. 180, putting D for A. I shall now prove by induction that if $Y=\frac{\cos x}{x}$, the coefficients of the same powers of l on each side are equal.

First, we will assume that

$$\left.\begin{aligned} a_n\frac{\cos x}{x} &= \frac{1}{n!}\frac{d^n}{dx^n}\frac{\cos x}{x} \\ a_{n-1}\frac{\cos x}{x} &= \frac{1}{(n-1)!}\frac{d^{n-1}}{dx^{n-1}}\frac{\cos x}{x} \end{aligned}\right\} \quad \ldots\ldots\ldots (5).$$

From (2)

$$[(n+1)^2 a_{n+1} + a_{n-1}]\frac{\cos x}{x} = -Da_n \frac{\cos x}{x}.$$

Putting for D its value from Art. 1, using (5), and putting d in place of $\frac{d}{dx}$ for shortness for a moment,

$$(n+1)^2 a_{n+1}\frac{\cos x}{x} = \left[-\frac{1}{(n-1)!}d^{n-1} - (xd^2+d+x)\frac{1}{n!}d^n\right]\frac{\cos x}{x}$$

$$= \left[-\frac{1}{n!}(nd^{n-1}+xd^n) - \frac{1}{n!}(xd^{n+2}+d^{n+1})\right]\frac{\cos x}{x} \ldots (6).$$

Now $$\frac{d^n}{dx^n}\left\{x \cdot \frac{\cos x}{x}\right\} = \cos\{x + \tfrac{1}{2}(n\pi)\}$$

and the left-hand side by Leibnitz's Theorem

$$= (xd^n + nd^{n-1})\frac{\cos x}{x}.$$

Thus the second part of the right-hand side of (6) may be written

$$\left[-\frac{1}{n!}\{xd^{n+2} + (n+2)d^{n+1}\} + \frac{n+2}{n!}d^{n+1} - \frac{1}{n!}d^{n+1}\right]\frac{\cos x}{x}.$$

Thus (6) becomes

$$(n+1)^2 a_{n+1}\frac{\cos x}{x} = -\frac{1}{n!}\cos\{x+\tfrac{1}{2}(n\pi)\} - \frac{1}{n!}\cos[x+\tfrac{1}{2}\{(n+2)\pi\}]$$

$$+ \frac{n+1}{n!}d^{n+1}\frac{\cos x}{x},$$

leaving finally

$$a_{n+1}\frac{\cos x}{x} = \frac{1}{(n+1)!}\frac{d^{n+1}}{dx^{n+1}}\frac{\cos x}{x}.$$

Thus, if the formulæ (5) hold for two successive values of n, they hold for the succeeding value. But by actual trial it can be readily shown that (5) hold when $n=1$ and $n=2$, therefore they hold universally.

Thus, if $x > l$, $Y = \frac{\cos x}{x}$ is the solution of (1).

3. If, instead of $\frac{\cos(x+l)}{x+l}$, we had $\frac{\sin(x+l)}{x+l}$ on the right-hand side of (1), we should find the solution of (1)

would be $Y=\frac{\sin x}{x}$, *whatever* value x have; for it can be shown, as I showed on p. 43, No. 123, that $\frac{\sin(x+l)}{x+l}$ can be expanded in a converging series of ascending powers of l, whether x be $>$ or $<l$. The remarkable thing is that this value of Y should be independent of l.

4. We proceed now to solve (1) by separation of symbols. Call the coefficient of Y, $F(D)$. Then, remembering that $F(D)$ is the product of an infinite number of real factors, the solution of (1) is

$$Y=\left\{\frac{1}{\frac{F'(A_1)}{D-A_1}}+\&c.\right\}\frac{\cos(x+l)}{x+l}.$$

This may be written

$$Y=\frac{x_1}{F'(A_1)}+\frac{x_2}{F'(A_2)}+\&c.\ldots\ldots\ldots\ldots(7),$$

where x_1 is determined by the equation

$$x\frac{d^2x_1}{dx^2}+\frac{dx_1}{dx}+xx_1-A_1x_1=\frac{\cos(x+l)}{x+l}\ldots\ldots\ldots(8).$$

If the solution of this equation could be obtained for all values of x and the proper values of A, and the results substituted in (7), it will appear from what follows that the solution so obtained, whilst apparently for *every* value of x a complicated function of x and l, has this property, that it is *really* a function of x and l only when $x<l$, and when $x>l$ it $=\frac{\cos x}{x}$.

(8), however, is not readily soluble for all values of x, so in what follows, instead of beginning with (1), I shall begin with an equation a little more complicated, but possessing the same peculiarities, for which, moreover, the equation corresponding to (8) is completely soluble, and which has the further advantage that it has an application in the Theory of Sound.

5. The equation is

$$-\frac{2}{\pi}\{\epsilon^{\frac{1}{2}\pi D}+\epsilon^{-\frac{1}{2}\pi D}\}\int_0^\infty\frac{d}{s^2}\sin\left(l\frac{d}{s}+D\phi\right)d\phi.\ Y=\frac{d}{dx}\frac{\cos(x+l)}{x+l}\ldots(9).$$

It will be at once observed that the coefficient of Y here

$$= \frac{dF(D)}{dl}.$$

It can be proved, by the method of Art. 2, that when $x > l$ the solution of (9) is

$$Y = \frac{\cos x}{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10);$$

in fact, this result can be at once obtained from what has gone before. For, suppose $x > l$, put for Y in (1) $\frac{\cos x}{x}$; (1) therefore becomes an identity. It may therefore be differentiated with regard to l. But the result of differentiating the right-hand side of (1) with regard to l is the same as if we differentiated it with regard to x. We thus get (10) as the solution of (9) when $x > l$.

6. We will now solve (9) by separation of symbols. Calling the coefficient of Y in (9) $\phi(D)$, the solution is

$$Y = \frac{x_1}{\phi'(A_1)} + \frac{x_2}{\phi'(A_2)} + \&c. \ldots\ldots\ldots\ldots (11),$$

where x_1, x_2, &c., are found from equations like

$$x\frac{d^2x_1}{dx^2} + \frac{dx_1}{dx} + xx_1 - A_1x_1 = \frac{d}{dx}\frac{\cos(x+l)}{x+l} \ldots (12),$$

A_1, A_2. &c. being the roots of the equation in A,

$$-\frac{2}{\pi}\left(\varepsilon^{\frac{1}{2}\pi A} + \varepsilon^{-\frac{1}{2}\pi A}\right)\int_0^\infty \frac{d}{s^{\frac{1}{2}}}\sin\left(l\frac{d}{s} + A\phi\right).d\phi = 0 \ldots (13),$$

which, on p. 183, No. 119, I have proved to be all real.

I have further proved on p. 41, No. 123, that (12) with the condition (13) is completely soluble for all values of x. I shall now prove that the solution (11) while, apparently for *all* values of x, a complicated function of x and l possesses the property, that it *is* a function of x and l only when $x < l$, and that when $x > l$ it actually reduces to $\frac{\cos x}{x}$. I think this is important to shew, in order to exclude the idea which we might easily entertain, that $Y = \frac{\cos x}{x}$ is a *particular* solution of (9).

7. Although the solution of (12) can be obtained for all values of x, yet the result is so complicated, that to effect this verification in the most general case would be tedious and perhaps unprofitable. I shall therefore examine in detail only the case where x is large compared with l. In this case, the solution of (12) can be obtained in a tolerably simple form. I have indicated how this is done on p. 44, No. 123, x_1 is the sum of an infinite number of terms like

$$\frac{a_n \sin x + b_n \cos x}{x^n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14),$$

where n has all values from 1 to ∞, and a_n, b_n are functions of A_1 and l. The series begins by converging, but ultimately diverges. The converging part, however, may be safely used. I shall retain only the first and most important term, but the reasoning similar to the following would apply to the other terms, if they are retained.

8. It will be found then that the solution of (12), confining oneself to the first term, is

$$x_1 = \frac{A_1 \sin(x+l) - \cos(x+l)}{x(1+A_1^2)} \quad \ldots\ldots\ldots (15).$$

Putting this and the corresponding values of x_2, &c., in (11), we get

$$Y = \frac{A_1 \sin(x+l) - \cos(x+l)}{x(1+A_1^2)\,\phi'(A_1)} + \&c. \quad \ldots\ldots\ldots (16).$$

We have now to sum the following series:

$$\frac{A_1}{(1+A_1^2)\,\phi'(A_1)} + \&c. \quad \text{and} \quad \frac{1}{(1+A_1^2)\,\phi'(A_1)} + \&c. \ldots (17),$$

A_1, A_2, &c., being derived from (13).

Since $\phi(A)$ is the product of an infinite number of real factors, which are all different, if $\chi(A)$ be any rational integral function of A, we have

$$\frac{\chi(A)}{\phi(A)} = \frac{\frac{\chi(A_1)}{\phi'(A_1)}}{A - A_1} + \&c. \quad \ldots\ldots\ldots\ldots (18).$$

In the particular case of the first term of (14), $\chi(A)$ must be taken $= 1$, but for higher terms it would be a function

of A. Putting then in (18) $\chi(A)=1$, and putting iA and $-iA$ for A in succession, we get

$$\frac{1}{\phi(iA)} = \frac{1}{\phi'(A_1)(iA-A_1)} + \&c.,$$

$$\frac{1}{\phi(-iA)} = \frac{-1}{\phi'(A_1)(iA+A_1)} + \&c.$$

Adding and subtracting,

$$\left.\begin{aligned} \frac{1}{\phi(iA)} + \frac{1}{\phi(-iA)} &= \frac{2A_1}{\phi'(A_1)(-A^2-A_1^2)} + \&c. \\ \frac{1}{\phi(iA)} - \frac{1}{\phi(-iA)} &= \frac{2iA_1}{\phi'(A_1)(-A^2-A_1^2)} + \&c. \end{aligned}\right\} \ldots\ldots(19).$$

These equations are true for all values of A, and therefore when $A=1$. We thus get the sums of the series (17).

9. We have now to put i for A in the left-hand side of (13). The result will be found to be indeterminate, but its value can be readily found from its expanded form given in (22) p. 180, No. 119, by putting i for A, l for x, and then differentiating each side with regard to l. It will be found that

$$\phi(i) = -\sin l - i\cos l;$$

in fact, it may be readily verified that when $A=i$, the solution of the equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + y(x+A) = 0 \ldots\ldots\ldots\ldots(20)$$

is $$y = \cos x - i\sin x.$$

It will thus be found from (19) that the sum of the first series in (17) $=\sin l$, and that the sum of the second series in (17) is $=-\cos l$. Put these values in (16), and it will obviously reduce to

$$Y = \frac{\cos x}{x},$$

as stated in Art. 6. If terms of a higher order were retained in (14), it could be proved by an exactly similar process that they vanish. We may remark that the definite integral solution of (20) becomes indeterminate, when for A we put any odd multiple of i, as we see from (7) p. 176, No. 119. It will be found that in these cases the solution of (20) is always finite.

NOTE ON ELASTICITY.

By *W. H. Besant, M.A., F.R.S.*

In Lamè's *Elasticité*, the expressions for the tangential and normal stresses are given in polar coordinates as the results of a transformation, which is somewhat laborious; and it may be useful to shew that they can be obtained by direct geometry.

If x, y, z be any three directions at right angles, and u, v, w the corresponding displacements,

$$N_1 = \lambda\theta + 2\mu\frac{du}{dx}, \quad T_1 = \mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right), \text{ \&c.}$$

Adopting Lamè's notation, Art. 83, consider separately the effects on the displacements U, V, W of the changes dr, $rd\phi$, and $r\cos\phi d\psi$.

Take first $dx = dr$, then, $V + \delta V$ and $W + \delta W$ being perpendicular to r, we at once have

$$\frac{du}{dx} = \frac{dU}{dr}, \quad \frac{dv}{dx} = \frac{dV}{dr}, \quad \frac{dw}{dx} = \frac{dW}{dr}.$$

Secondly, take $dy = rd\phi$, then (fig. 11)

$$\frac{du}{dy} = \frac{(U+\delta U)\cos\delta\phi - (V+\delta V)\sin\delta\phi - U}{r\delta\phi}$$

$$= \frac{dU}{rd\phi} - \frac{V}{r},$$

$$\frac{dv}{dy} = \frac{(V+\delta V)\cos\delta\phi + (U+\delta U)\sin\delta\phi - V}{r\delta\phi} = \frac{dV}{rd\phi} + \frac{U}{r},$$

and $$\frac{dw}{dy} = \frac{dW}{rd\phi}.$$

Thirdly, take $\delta z = r\cos\phi\delta\psi$,

then, (fig. 12), $\delta\theta = POP' = \dfrac{r\cos\phi\delta\psi}{r} = \cos\phi\delta\psi$,

and $$\delta\theta' = PTP' = \frac{PP'}{PT} = \frac{r\cos\phi d\psi}{r\cot\phi} = \sin\phi\delta\psi.$$

We now see that

$$\frac{du}{dz} = \frac{1}{r\cos\phi\delta\psi}\{(U+\delta U)\cos\delta\theta - (W+\delta W)\sin\delta\psi\cos\phi - U\},$$

observing that $V+\delta V$ is in the perpendicular plane

$$= \frac{dU}{r\cos\phi d\psi} - \frac{W}{r},$$

$$\frac{dv}{dz} = \frac{1}{r\cos\phi\delta\psi}\{(V+\delta V)\cos\delta\theta' + (W+\delta W)\delta\psi\sin\phi - V\}$$

$$= \frac{dV}{r\cos\phi d\psi} + \frac{W}{r}\tan\phi.$$

Lastly, (fig. 13),

$$\frac{dv}{dz} = \frac{1}{r\cos\phi\delta\psi}\{(W+\delta W)\cos\delta\psi + (U+\delta U)\cos\phi\delta\psi - (V+\delta V)\sin\phi\delta\psi - W\}$$

$$= \frac{dW}{r\cos\phi d\psi} + \frac{U}{rd\psi} - \frac{V}{r}\tan\phi.$$

Substituting in the equations for N and T, the expressions given by Lamè, Art. 84, follow at once.

ON A PROPERTY OF PERSYMMETRIC DETERMINANTS.

By *Thomas Muir, M.A., F.R.S.E.*

1. The only property which is usually given as special to the persymmetric determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots a_n \\ a_2 & a_3 & a_4 & \dots a_{n+1} \\ a_3 & a_4 & a_5 & \dots a_{n+2} \\ \dots & \dots & \dots & \dots \\ a_n & a_{n+1} & a_{n+2} & \dots a_{2n-1} \end{vmatrix}$$

is that it remains unaltered in value when $a_1, a_2, a_3, \dots a_{2n-1}$ are replaced by the 0^{th}, 1^{st}, 2^{nd}, $\dots$ $(2n-2)^{\text{th}}$ differences of a_1 viewed as the first term of the series $a_1, a_2, a_3, \dots a_{2n-1}$.

This theorem is obtained in two stages: (1°) by going backwards from the last row and diminishing each element of each row by the corresponding element of the row before it, then treating the $n-1$ resulting rows as the previous n were treated, then in the same way the $n-2$ rows resulting from this second set of operations, and so on as far as possible; and (2°) by repeating in order on the determinant now reached the whole of the previous operations, but dealing with columns instead of rows. The total effect produced in the first stage upon any element of the last row is to diminish it by $n-1$ times the corresponding element of the $(n-1)^{\text{th}}$ row, increase it by $\frac{1}{2}(n-1)(n-2)$ times the corresponding element of the $(n-2)^{\text{th}}$ row, diminish it by $\frac{1}{6}(n-1)(n-2)(n-3)$ times the corresponding element of the $(n-3)^{\text{th}}$ row, and so on; and the total effect upon any element of any other row is similar. It is thus seen that what is effected in the first stage is exactly what would be effected by the multiplication of the original determinant by the determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ -1 & 3 & -3 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (-1)^{n-1} & (-1)^{n-2}C_{n-1,1} & (-1)^{n-3}C_{n-1,2} & (-1)^{n-4}C_{n-1,3} & \dots & -C_{n-1,1} & 1 \end{vmatrix},$$

where the non-zero elements of the rows are the coefficients of $(-a+b)^r$ for $r=0, 1, 2, \ldots(n-1)$. Hence the transformation, which is the subject of the theorem we started with, is the result of the multiplication of the persymmetric determinant by the second power of the determinant just written.

2. Let us now consider the result of multiplying the persymmetric determinant by the second power of

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \ldots 0 & 0 \\ m & 1 & 0 & 0 & \ldots 0 & 0 \\ m^2 & 2m & 1 & 0 & \ldots 0 & 0 \\ m^3 & 3m^2 & 3m & 1 & \ldots 0 & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ m^{n-1} & C_{n-1,1}m^{n-2} & C_{n-1,2}m^{n-3} & C_{n-1,3}m^{n-4} & \ldots C_{n-1,1} & 1 \end{vmatrix},$$

denoting the determinant produced after one multiplication by $|\gamma_{1n}|$ and the final determinant by $|E_{1n}|$. Then any element of the final determinant

$$\begin{aligned} E_{rs} = {} & m^{r-1}\gamma_{s,1} + C_{r-1,1}m^{r-2}\gamma_{s,2} + C_{r-1,2}m^{r-3}\gamma_{s,3} + \ldots \\ = {} & m^{r-1}\{m^{s-1}a_1 + C_{s-1,1}m^{s-2}a_2 + C_{s-1,2}m^{s-3}a_3 + \ldots\} \\ & + C_{r-1,1}m^{r-2}\{m^{s-1}a_2 + C_{s-1,1}m^{s-2}a_3 + C_{s-1,2}m^{s-3}a_4 + \ldots\} \\ & + C_{r-1,2}m^{r-3}\{m^{s-1}a_3 + C_{s-1,1}m^{s-2}a_4 + C_{s-1,2}m^{s-3}a_5 + \ldots\} \\ & + \ldots\ldots\ldots\

The right-hand member here is constant so long as $r+s$ is constant; therefore

$$E_{rs} = E_{rtk,\ stk};$$

that is to say, the determinant $|E_{in}|$ is persymmetric, and we have thus the generalised theorem:

The persymmetric determinant of $a_1, a_2, a_3, \ldots a_{2n-1}$ *is equal to the persymmetric determinant of*

$$a_1,\ (a_1, a_2 \text{X} m, 1),\ (a_1, a_2, a_3 \text{X} m, 1)^2,\ \ldots (a_1, a_2, a_3, \ldots a_{2n-1} \text{X} m, 1)^{2n-2}.$$

The case where $m=-1$ is that originally started with, and the mode of proof there sketched is of course also applicable here, but for various reasons it is not so satisfactory as that which has been given.

Beechcroft, Bishopton, N.B.
15th August, 1881.

ON THE NUMBER OF SYSTEMS OF PLANE EQUIPOTENTIAL LINES OF THE SECOND DEGREE, SYMMETRICAL WITH RESPECT TO A FIXED LINE.

By *W. M. Hicks, M.A.*

I PROPOSE to investigate the number and nature of those families of surfaces of the second degree, which are symmetrical about a line (the axis of y say), and which can serve for equipotential surfaces. The method will apply for all cases, either of non symmetry, or for higher degrees, but the working out would be very laborious.

The general question may be thus stated: Given a functional relation $f(z, w)=0$ between two complexes $z=x+yi$ and $w=\phi+\psi i$, what are the conditions that, if we write $f(z, w)=P+Qi$, P, Q being real, the result of eliminating ψ between $P=0$ and $Q=0$ may be expressible as an algebraical function of x, y of the n^{th} degree. The method I employ in the following is different from this. I start by supposing the resultant given in the form required, the coefficients being supposed functions of the potential ϕ, which is a parameter of the family of curves. The coefficients are then determined from the condition that $\nabla^2\phi$ must vanish

identically. This gives a number of differential equations to determine the coefficients greater than the number of unknowns. But it is easy to see that by a proper choice of constants, these can all be satisfied simultaneously in at least two ways.

Taking then the particular case considered in this paper, any curve of the second degree symmetrical with respect to the axis of y can be expressed by

$$x^2 + by^2 + 2fy + c = 0 \equiv F \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If b, f, c be regarded as functions of ϕ, the condition that $\nabla^2\phi = 0$ is

$$F_{xx} + F_{yy} + 2F_{x\phi}\phi_x + 2F_{y\phi}\phi_y + F_{\phi\phi}(\phi_x^2 + \phi_y^2) = 0,$$

or
$$F_\phi^2(F_{xx} + F_{yy}) - 2F_yF_\phi F_{y\phi} + F_{\phi\phi}(F_x^2 + F_y^2) = 0,$$

identically for all values of (x, y) satisfying the relation (1). Substituting their values for F_x, F_y, &c., and denoting differential coefficients with respect to ϕ by dashed letters, this becomes

$$(1+b)(b'y^2 + 2f'y + c')^2 - 4(by+f)(b'y+f')(b'y^2 + 2f'y + c')$$
$$+ 2\{x^2 + (by+f)^2\}(b''y^2 + 2f''y + c'') = 0.$$

Substituting for x^2 from (1), and combining the first two terms,

$$(b'y^2+2f'y+c')[b'(1-3b)y^2+2\{(1-b)f'-2b'f\}y+c'(1+b)-4ff'] + 2\{b(b-1)y^2 + 2f(b-1)y + f^2 - c\}(b''y^2 + 2f''y + c'') = 0,$$

an expression of the fourth degree which has to vanish identically. This gives five equations to determine the three quantities b, f, c. Equating to zero the coefficients of the different powers of y, we get, beginning with that of y^4,

$$\left.\begin{aligned}
&2b(b-1)b'' + b'^2(1-3b) = 0\\
&(b-1)fb'' + b(b-1)f'' + (1-2b)b'f' - fb'^2 = 0\\
&b''(f^2-c) + 4ff''(b-1) + b(b-1)c''\\
&\qquad + b'c'(1-b) + 2(1-b)f'^2 - 6b'ff' = 0\\
&f''(f^2-c) + f(b-1)c'' + f'c' - fb'c' - 2ff'^2 = 0\\
&2c''(f^2-c) + (1+b)c'^2 - 4ff'c' = 0
\end{aligned}\right\} \ldots(2).$$

From the first of these

$$\frac{2b''}{b'} = \frac{3b-1}{b(b-1)}\,b',$$

$$\frac{b'}{\sqrt{b(b-1)}} = -\alpha,$$

$$\log\frac{\sqrt{b}+1}{\sqrt{b}-1} = \alpha\phi+\beta.$$

Now ϕ being a potential so also is $\alpha\phi+\beta$, we may therefore put ϕ for $\alpha\phi+\beta$, provided we also take into account $\alpha=0$, or $b=$ const. There are then two cases. (1) $b=$ const. and (2) $b=\coth^2\phi$.

Case I. $b=$ const. The four last equations of (2) now become

$$\left.\begin{aligned} &b(b-1)f''=0\\ &(b-1)(4ff''+bc''-2f'^2)=0\\ &f''(f^2-c)+f(b-1)c''+ff'c'-2ff'^2=0\\ &2c''(f^2-c)+(1+b)c'^2-4ff'c'=0 \end{aligned}\right\}\ldots(3).$$

The first gives three sub-cases, viz. (α) $b=0$, (β) $b=1$, and (γ) $f''=0$.

(α). $b=0$. The three last equations become

$$\left.\begin{aligned} &2ff''-f'^2=0\\ &f''(f^2-c)-fc''+f'c'-2ff'^2=0\\ &2c''(f^2-c)+c'^2-4ff'c'=0 \end{aligned}\right\}.$$

The first of these gives $\sqrt{f}=\alpha\phi+\beta$, or from the remark made above $f=\phi^2$ or const.

If f is constant c must satisfy the two equations $c''=0$ and $2c''(f^2-c)+c'^2=0$, *i.e.* c is constant.

This gives no family of curves.

If $f=\phi^2$, c must satisfy the equations

$$\left.\begin{aligned} &\phi^2c''-2\phi c'+2c=-6\phi^4\\ &2(\phi^4-c)c''+c'(c'-8\phi^3)=0 \end{aligned}\right\}.$$

From the first $c=A\phi+B\phi^2-\phi^4$, which satisfies the second only for $A=B=0$.

The curves corresponding to this subcase are

$$x^2+2\phi^2y-\phi^4=0;$$

a series of confocal parabolas.

(β). $b=1$. The three last equations of (3) become

$$\left.\begin{aligned} 0&=0\\ f''(f^2-c)+f'c'-2ff'^2&=0\\ c''(f^2-c)+c'^2-2ff'c'&=0 \end{aligned}\right\}.$$

These are

$$\left.\begin{aligned} f''(f^2-c)-f'\frac{d}{d\phi}(f^2-c)&=0\\ c''(f^2-c)-c'\frac{d}{d\phi}(f^2-c)&=0 \end{aligned}\right\},$$

whence $$\frac{f'}{f^2-c}=\alpha,\quad \frac{c'}{f^2-c}=\beta,$$

therefore $$f=\frac{\alpha}{\beta}c+\text{const.}=\frac{\alpha c+\gamma}{\beta},$$

$$\frac{c'}{\left(\frac{\alpha}{\beta}c+\frac{\gamma}{\beta}\right)^2-c}=\beta,$$

$$\frac{c'}{\left(c+\frac{2\alpha\gamma-\beta^2}{2\alpha^2}\right)^2+\frac{\beta^2(4\alpha\gamma-\beta^2)}{4\alpha^4}}=\frac{\alpha^2}{\beta}.$$

There are here again two cases according as $\beta^2 \lesseqgtr 4\alpha\gamma$. If $\beta^2>4\alpha\gamma$, the integral is

$$\frac{\alpha^2}{\beta\sqrt{(\beta^2-4\alpha\gamma)}}\log\frac{2\alpha^2c-(\beta^2-2\alpha\gamma)-\beta\sqrt{(\beta^2-4\alpha\gamma)}}{2\alpha^2c-(\beta^2-2\alpha\gamma)+\beta\sqrt{(\beta^2-4\alpha\gamma)}}=\frac{\alpha^2}{\beta}\phi+\text{const.}$$

or by a proper choice of the constants in ϕ

$$\frac{2\alpha^2c-(\beta^2-2\alpha\gamma)}{\beta\sqrt{(\beta^2-4\alpha\gamma)}}=\coth\phi;$$

whence $$2\alpha^2c=\beta^2-2\alpha\gamma+\beta\sqrt{(\beta^2-4\alpha\gamma)}\coth\phi,$$

$$2\alpha f=\beta+\sqrt{(\beta^2-4\alpha\gamma)}\coth\phi.$$

Put $$\beta+\sqrt{(\beta^2-4\alpha\gamma)}=2h,\quad \beta-\sqrt{(\beta^2-4\alpha\gamma)}=2k.$$

Then $$2\alpha^2c=h^2+k^2+(h^2-k^2)\coth\phi,$$

$$2\alpha f=h+k+(h-k)\coth\phi.$$

But a relation must hold between h, k, for if these values be substituted in either of the equations, they will not be satisfied unless $h^2 - k^2 = 0$, $h = k$ gives no family of curves, $k = -h$ gives

$$\alpha^2 c = h^2,$$

$$\alpha f = h \coth \phi,$$

and the family of curves is

$$x^2 + y^2 + \frac{2h}{\alpha} y \coth \phi + \frac{h^2}{\alpha^2} = 0,$$

that is *the series of circles having the same two points as centres of inversion*, viz., $\left(0, \pm \frac{h}{\alpha}\right)$.

If $\beta^2 < 4\alpha\gamma$ the integral is

$$\frac{2\alpha^2}{\beta \sqrt{(4\alpha\gamma - \beta^2)}} \tan^{-1} \frac{2\alpha^2 c + 2\alpha\gamma - \beta^2}{\beta \sqrt{(4\alpha\gamma - \beta^2)}} = \frac{\alpha^2}{\beta} \phi + \text{const.},$$

and as before we may choose ϕ so that

$$2\alpha^2 c = \beta^2 - 2\alpha\gamma + \beta \sqrt{(4\alpha\gamma - \beta^2)} \tan \phi$$

$$= 2hk + (h^2 - k^2) \tan \phi,$$

and $$2\alpha f = h + k + (h - k) \tan \phi.$$

Introducing the condition $k = -h$, we have

$$\alpha^2 c = -h^2,$$

$$\alpha f = h \tan \phi,$$

and the curves are

$$x^2 + y^2 + \frac{2h}{\alpha} y \tan \phi - \frac{h^2}{\alpha^2} = 0,$$

that is *the series of circles through the two fixed points* $\left(\pm \frac{h}{\alpha}, 0\right)$, *orthogonal to the former series.*

(γ). $f'' = 0$ whence, $f = \text{const.}$ or ϕ.
If $f = \text{const.}$ the three last equations of (3) become

$$\left.\begin{aligned} b(b-1)c'' &= 0 \\ f(b-1)c'' &= 0 \\ 2c''(f^2 - c) + (1+b)c'^2 &= 0 \end{aligned}\right\}.$$

With $b=0$ we get $f=0$, $c=\phi^2$, and the curves are straight lines parallel to the axis of y (i.e. a linear family). With $b=1$, $c''(f'^2-c)+c'^2=0$, whence $c=f'^2-e^{-\phi}$.

The curves now are

$$x^2+y^2+2fy+f'^2-e^{-\phi}=0,$$

a series of concentric circles about the point $(0.-f)$, *and* $\phi=\log r$.

With $c''=0$ the last equation gives $(1+b)c'^2=0$.

If $c=\phi$, then $b=-1$.

If $c=$ const, then b is arbitrary.

The former gives

$$x^2-y^2+2fy+\phi=0,$$

a series of coaxal rectangular hyperbolas.

The latter gives a single curve only.

If $f=\phi$ the equations are

$$\left.\begin{aligned}(b-1)(bc''-2)&=0\\(b-1)\phi c''+c'-2\phi&=0\\2c''(\phi^2-c)+(1+b)c'^2-4\phi c'&=0\end{aligned}\right\}.$$

If $b=1$, the solution is $c=\phi^2$, which gives

$$x^2+y^2+2\phi y+\phi^2=0,$$

a nugatory result.

If $$c''=\frac{2}{b},\quad c=\frac{\phi^2}{b}+\alpha\phi+\beta.$$

The second equation then requires $\alpha=0$, and last requires $\beta=0$ and $b=1$, which, therefore, returns to the other.

Case II. $b=\coth^2\phi$. If in the second equation of (2) we put $f=b$ the equation is satisfied, when b has the value given by the first, viz. $\coth^2\phi$. The equation becomes

$$\coth^2\phi.f''-2\coth\phi(1-2\coth^2\phi)f'+2\operatorname{cosech}^4\phi.f=0.$$

Assume $f=A\coth^n\phi$, the equation is satisfied if

$$(n-1)(n-2)=0.$$

Hence, the solution is

$$f=A\coth\phi+B\coth^2\phi.$$

The last term of this is the same as the coefficient of y

If then we alter the origin it is possible to get rid of it. We can therefore put $B=0$, and then

$$f=A\coth\phi.$$

The three last equations to be satisfied are then

$$\cosh^2\phi.c''+2\coth\phi.c'-2(1+2\cosh^2\phi)(c-A^2)=0,$$

$$A\left\{c''+\frac{\cosh^2\phi+1}{\sinh\phi\cosh\phi}c'-2c+2A^2\right\}=0,$$

$$2(A^2\coth^2\phi-c)c''+(1+\coth^2\phi)c'^2+4A^2\coth\phi\,\mathrm{cosech}^2\phi c'=0.$$

If in the second equation we regard A as not equal to zero, and eliminate $c''.c'$ between the first and second, we get

$$\frac{2\cosh^2\phi+1}{\cosh^2\phi+1}\coth\phi(c-A^2)=0,$$

which can only be satisfied by $c=A^2$. This also satisfies the last equation, and the curves are

$$x^2+(y\coth\phi+A)^2=0,$$

which again is nugatory.

If A is zero, i.e. $f=0$, the second is satisfied and the others are

$$\cosh^2\phi.c''+2\coth\phi.c'-2(1+2\cosh^2\phi)c=0,$$

$$2cc''-(1+\coth^2\phi)c'^2=0.$$

The solution of the last is $c=B+A\cosh^2\phi$. This satisfies the former if $B=0$.

The curves may now be written

$$\frac{x^2}{a^2\cosh^2\phi}+\frac{y^2}{a^2\sinh^2\phi}=1,$$

which are *confocal ellipses.*

Returning we notice that originally we replaced $\alpha\phi+\beta$ by ϕ, regarding α as real. But being an arbitrary constant it may be imaginary. We may therefore put ϕi for ϕ, and since

$$\cosh\phi i=\cos\phi,\quad \sinh\phi i=-i\sin\phi,$$

we get, in the same way as before,

$$\frac{x^2}{a^2\cos^2\phi}-\frac{y^2}{a^2\sin^2\phi}=1,$$

which are *confocal hyperbolas.*

Collecting the results of the investigation we find that the only curves of the second degree, symmetrical about a line, which can serve as equipotential surfaces, are

1. Concentric circles, Case I., γ. 2.
2. Confocal conics, I., α2. and II2.
3. Rectangular hyperbola, I., γ 3.
4. Circles through two fixed points, I., β 2.
5. The system of circles orthogonal to 4, I. β1.

A similar method applied to surfaces symmetrical with respect to three perpendicular planes easily leads to (1) concentric spheres, (2) confocal quadrics with their limits of confocal cones, and (3) the hyperboloids of the form $x^2 + y^2 - 2z^2 = \phi$, which are the analogue of (3) above.

MATHEMATICAL NOTES.

A proof of the parallelogram of forces.

The method of proof is as follows:

These axioms are assumed—

(*a*) The line of action of the resultant of two forces acting at a point passes through that point.

(*b*) The resultant of two equal forces acting at a point bisects the angle between them.

(*c*) The principle of the 'Transmissibility of force.'

Starting from these axioms, and taking the usual definitions of 'like couples,' 'arm of a couple,' 'moment of a couple,' 'moment of a force about a point,' all of which ideas are entirely independent of the parallelogram of forces, it is proved that two like couples of equal moment in the same plane are equivalent to one another. From this Varignon's theorem of moments is deduced, and thence the parallelogram of forces easily follows.

The proof may be divided into three propositions:

I. Two like couples of equal moment are equivalent to one another.

II. The algebraical sum of the moments about any point of a system of forces in equilibrium is zero.

III. The proposition known as the parallelogram of forces.

(In the above statements all points, forces, &c., are supposed to be in one plane.)

PROP. I. Consider the couple consisting of the forces P, P, acting at the extremities of the arm AB (fig. 14).

Draw AB' equal to AB, making any angle with it.

Draw $B'O$ at right angles to AB', meeting the perpendicular to AB through B in O.

At B' apply forces P, P, along $B'O$ and OB'; also at A forces P, P, in the directions $B'O$, OB'.

These forces will not alter the effect of the original couple.

By axiom (b), the resultant of two of the three forces at A is along AO, while the resultant of the two forces whose lines of application meet in O is equal to the same and is along OA, so that there is left a couple like the original one, of equal moment, whose arm is AB'.

By turning the arm AB about either extremity alternately, it is clear it can be brought into any other position whatsoever.

Now (fig. 15) let AB be divided into n equal parts in the points C_1, C_2, C_3,... C_{n-1}.

The effect of the original couple is unaltered if the forces P, P, acting in opposite directions at right angles to AB be applied at all the points C_1, C_2,... C_{n-1}.

We have now n like couples, whose moments and arms are equal, viz.

P, P, at the extremities of the arm AC,
..... C_1C_2,
.....,
..... $C_{-1}B$.

But by the previous part of this proof, each couple can be moved without altering its effect, so as to have its arm coincident with AC_1: so that the couple P, P, arm a, is equivalent to a like couple, nP, nP, arm $\frac{a}{n}$.

Hence two like couples of equal moment, and with commensurable arms are equivalent to one another: and as incommensurable quantities can be made to differ from commensurable ones by quantities as small as we please, the same applies to like couples of equal moment with incommensurable arms; or of course a 'reductio ad absurdum' proof for couples with incommensurable arms could be easily given.

Corollary. Any number of couples then are equivalent to a single resultant couple whose moment is equal to the algebraical sum of their moments.

PROP. II. Let P, Q, R, be a system of forces in equilibrium, O any point.

The equilibrium is not disturbed if at O are applied pairs of equal and opposite forces, equal to P, Q, and in directions respectively parallel to those of P, Q.

The system now consists of a number of forces acting at O, and a number of couples, whose moments are equal to the moments about O of the coresponding original forces.

If the effect of these couples be not nil, they are equivalent to a single resultant couple whose moment is equal to the algebraical sum of their moments; and one end of its arm may be moved to O, so that the system reduces to a number of forces at O, and a force which does not pass through O, which by axiom (a) clearly could not maintain equilibrium.

The algebraical sum of the moments of the couples, *i.e.* the algebraical sum of the moments about O of the original forces, must therefore be zero.

Corollary. Also, as any one of a system of forces in equilibrium is equal and opposite to the resultant of the rest, the algebraical sum of the moments of any number of forces about any point is equal to the moment of their resultant about that point.

PROP. III. Let AB, AC (fig. 16) represent two forces acting at A.

Complete the parallelogram $ACDB$, and join AD.

The algebraical sum of the moments of AB, AC about D is zero, and therefore their resultant acts along AD.

The algebraical sum of the moments of AB, AC about B is equal to the moment of AD about B, whence it follows that AD represents the magnitude of the resultant.

The resultant of two parallel forces is easily found as follows:

The algebraical sum of the moments of two parallel forces P and Q is zero about any point in a line whose distances from their lines of action are inversely proportional to the magnitudes of P and Q, provided that this line be between the lines of action, if the forces act in the same direction, but not between them if they act in opposite directions. This line then must be the line of action of their resultant of P and Q.

Also by taking moments about any point in the line of action of either force we see that the resultant is equal to the sum of the two forces if they act in the same direction, to their difference if they act in opposite directions.

J. GREAVES.

Elementary proof of the contact of the nine-point circle with the inscribed and escribed circles.

I. (Figs. 17 and 18) DE a chord of the circle DEF touches the circle GHP at G. ACF and OG are radii of the circles perpendicular to DE. *Then, if rect.* $DG.GE = 2$ *rect.* $OG.CF$, *the circles touch one another.* [This is the converse of part of the general proposition that "*the difference of the squares on the tangents to two circles from any point is double the rectangle under the distance between their centres and the distance of the given point from the radical axis.*"]

It may be proved in an elementary manner thus:

Let FG cut GHP in P, and OR be perpendicular to FP. Then RP touches GHP and is equal to RG. Draw RT touching DEF.

Then (by hyp.) $EG.GD = 2OG.FC$

$$= 2CG.GR$$

(similar triangles RGO, FCG),

whence $CE^2 + GR^2 = CR^2$ (Euc. II. 5, 4 or Euc. II. 6, 7),

whence $GR^2 = DR.RE$ (Euc. II. 6)

$$= RT^2,$$

and therefore RP, RT, RG are all equal.

Then (fig. 17)

$$2 \angle TPG = \angle TRG = \angle TAF,$$

and (fig. 18) $2 \angle TPG = \angle TRG = TAF = 2TEF$,

whence $TEF + TPF = 2$ right angles.

Hence in both figures P is on circumference of DEC, and the circles touch at P.

II. Now let ABC (fig. 19) be any* triangle; O, O', Q, O_1 the centres of the circumscribed, inscribed, nine-point, and escribed (opposite A) circles respectively. $VODW$ is the diameter of the circumscribed circle bisecting BC; $PQXR$ is that of the nine-point circle perpendicular to BC; it cuts AW (which passes through O' and O_1) at Y; $O'H$, $O'G$ are radii of inscribed circle perpendicular to AB and BC. GO' produced meets VB at S. SZ, $O'K$ are drawn parallel to, and AKL perpendicular to BC (Z being on VW). Then it may be easily shown that

$VS = VA$, DP is parallel to WA, $DR = OY = \frac{1}{2}VA$.

* In describing the construction I have supposed the angle A acute, and $B > C$.

Then we have

$$O'K : O'A :: XR : RD$$
(similar triangles $AO'K$, DRX)
$$:: 2XR : AV,$$
$$O'A : O'H :: VS : SZ$$
(similar triangles $AO'H$, VSZ)
$$:: AV : DG,$$

whence *ex æq.* rect. $LG.GD = 2$ rect. $XR.O'G$.

Similarly, if G_1 be the point at which the escribed circle touches BC, it may be shown that

$$\text{rect. } LG_1.G_1D = 2 \text{ rect. } XR.O_1G_1,$$

whence the nine-point circle touches both the inscribed and escribed circles. W. F. McMICHAEL.

A system of differential formulæ in Elliptic Functions. On p. 112 of the *Fundamenta Nova* Jacobi uses the formula

$$\frac{d^2}{du^2} \log \operatorname{sn} u = k^2 \operatorname{sn}^2 u - \frac{1}{\operatorname{sn}^2 u}$$

to deduce the q-series for $\frac{1}{\operatorname{sn}^2 u}$ from the series for $\operatorname{sn}^2 u$ and $\log \operatorname{sn} u$. It seems worth while to remark with regard to this formula that there are two corresponding results involving $\operatorname{cn} u$ and $\operatorname{dn} u$, and to notice a certain form in which this system of equations may be written.

The other formulæ are

$$\frac{d^2}{du^2} \log \operatorname{cn} u = -k^2 \operatorname{cn}^2 u - \frac{k'^2}{\operatorname{cn}^2 u},$$

$$\frac{d^2}{du^2} \log \operatorname{dn} u = - \operatorname{dn}^2 u + \frac{k'^2}{\operatorname{dn}^2 u},$$

and we have therefore

$$\frac{d^2}{du^2} \log \operatorname{sn} u = k^2 \{\operatorname{sn}^2 u - \operatorname{sn}^2 (u + iK')\},$$

$$\frac{d^2}{du^2} \log \operatorname{cn} u = -k^2 \{\operatorname{cn}^2 u - \operatorname{cn}^2 (u + K + iK')\},$$

$$\frac{d^2}{du^2} \log \operatorname{dn} u = - \{\operatorname{dn}^2 u - \operatorname{dn}^2 (u + K)\}.$$

Expressing the right-hand members of these equations in terms of sn's only, we find

$$\frac{d^2}{du^2}\log \operatorname{sn} u = k^2\{\operatorname{sn}^2 u - \operatorname{sn}^2(u+iK')\},$$

$$\frac{d^2}{du^2}\log \operatorname{cn} u = k^2\{\operatorname{sn}^2 u - \operatorname{sn}^2(u+K+iK')\},$$

$$\frac{d^2}{du^2}\log \operatorname{dn} u = k^2\{\operatorname{sn}^2 u - \operatorname{sn}^2(u+K)\},$$

and therefore also, by subtraction,

$$\frac{d^2}{du^2}\log \frac{\operatorname{sn} u}{\operatorname{cn} u} = k^2\{\operatorname{sn}^2(u+K+iK') - \operatorname{sn}^2(u+iK')\},$$

$$\frac{d^2}{du^2}\log \frac{\operatorname{sn} u}{\operatorname{dn} u} = k^2\{\operatorname{sn}^2(u+K) - \operatorname{sn}^2(u+iK')\},$$

$$\frac{d^2}{du^2}\log \frac{\operatorname{cn} u}{\operatorname{dn} u} = k^2\{\operatorname{sn}^2(u+K) - \operatorname{sn}^2(u+K+iK')\},$$

so that the second differential coefficients of the logarithms of the twelve primary elliptic functions, viz.

$$\operatorname{sn} u,\quad \operatorname{cn} u,\quad \operatorname{dn} u,\quad \frac{\operatorname{sn} u}{\operatorname{cn} u},\quad \frac{\operatorname{sn} u}{\operatorname{dn} u},\quad \frac{\operatorname{cn} u}{\operatorname{dn} u},$$

and their reciprocals, may all be expressed in the form

$$k^2(\operatorname{sn}^2 X - \operatorname{sn}^2 Y),$$

where X and Y denote two of the four quantities u, $u+K$, $u+iK'$, $u+K+iK'$.

J. W. L. Glaisher.

A theorem on the attraction of confocal ellipsoids.

It is easily seen that

1. An ellipsoid may be transformed by a pure strain into a confocal ellipsoid passing through an arbitrary point.

2. Any point on one of the ellipsoids, and the point into which it is transformed, are corresponding points on the two ellipsoids.

3. The component attraction of a finite straight line AB parallel to its own direction, upon an external point P of unit mass, is

$$\int_{AP}^{BP} k\rho f(r)\,dr = k\rho\{F(BP) - F(AP)\},$$

where ρ is the density of the line, k its cross section, and $f(r)$ determines the law of force.

Divide the original ellipsoid into elementary strips by planes parallel to the planes of xy and xz, then after the strain, if the ellipsoid be referred to the principal axes, these planes will still remain planes, and will be parallel to their original positions.

Let Q_1, Q_1' (fig. 20) be the ends of such a strip, Q_2, Q_2' the ends of the transformed strip. Then Q_1, Q_2, and Q_1', Q_2' are corresponding points. Let P_1, P_2 be any other pair of corresponding points.

Let ρ_1, ρ_2 be the densities of the two ellipsoids, and we will assume that ρ_1 is constant, from which it will follow that ρ_2 is constant also.

Let k_1, k_2 be the cross sections of the strips.

Let a_1, b_1, c_1; a_2, b_2, c_2 be the semi-axes of the two ellipsoids, and let

$$a_2 = \lambda a_1,\ b_2 = \mu b_1,\ c_2 = \nu c_1 :$$

only one of the three quantities λ, μ, ν is arbitrary, as two of them are determined by the condition that the ellipsoids are to be confocal.

Finally, let x_1, y_1, z_1; x_2, y_2, z_2 be the coordinates of any point in the ellipsoid before and after the strain respectively.

Now corresponding strips in the two ellipsoids consist of the same portion of matter, so that they have the same mass, which gives us

$$\rho_2 dx_2 dy_2 dz_2 = \rho_1 dx_1 dy_1 dz_1,$$

or

$$\frac{\rho_2}{\rho_1} = \frac{dx_1 dy_1 dz_1}{dx_2 dy_2 dz_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (4).$$

Again,

$$\frac{\text{attraction of strip } Q_2Q_2' \text{ on } P_1}{\text{attraction of strip } Q_1Q_1' \text{ on } P_2} = \frac{k_2\rho_2\{F(Q_2'P_1) - F(Q_2P_1)\}}{k_1\rho_1\{F(Q_1'P_2) - F(Q_1P_2)\}}$$

$$= \frac{k_2\rho_2}{k_1\rho_1} \text{ (by 2)} = \frac{\rho_2 dy_2 dz_2}{\rho_1 dy_1 dz_1} = \frac{dx_1}{dx_2} \text{ (by 4)} = \frac{1}{\lambda} = \frac{a_1}{a_2}.$$

Hence, summing for all the strips, we get the theorem:—

If a homogeneous ellipsoid be transformed by a pure strain into a confocal ellipsoid, the attraction of the first ellipsoid at a point on the surface of the second ellipsoid, resolved parallel to an axis is to the attraction of the second ellipsoid at the corresponding point on the surface of the first ellipsoid so resolved, as that axis of the second ellipsoid is to the corresponding axis of the first ellipsoid.

G. W. VON TUNZELMANN.

ON ELLIPTIC FUNCTIONS.

I.

By *J. W. L. Glaisher.*

Introduction, §§ 1–4.

§1. The elliptic functions $\sin\mathrm{am}\,u$, $\cos\mathrm{am}\,u$, $\Delta\mathrm{am}\,u$ first make their appearance in the *Fundamenta Nova* (§ 17, pp. 30, 31) in the following manner.

Putting $$\int_0^{\phi} \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}} = u,$$

Jacobi expresses ϕ as a function of u by the equation

$$\phi = \mathrm{am}\,u,$$

and remarks that it therefore follows that if

$$\int_0^{x} \frac{dx}{\sqrt{\{(1-x^2)\,(1-k^2x^2)\}}} = u,$$

then $$x = \sin\mathrm{am}\,u.$$

After defining K he immediately defines $K-u$ as the complement of u, and denotes the amplitude of the complement by the functional sign coam, so that

$$\mathrm{am}\,(K-u) = \mathrm{coam}\,u.$$

In accordance with the meaning assigned by Legendre to the symbol Δ, the function $\Delta\mathrm{am}\,u$ is then defined by the equation

$$\Delta\mathrm{am}\,u = \sqrt{(1-k^2\sin^2\mathrm{am}\,u)},$$

and Jacobi proceeds, "Ipsas expressiones $\sin\mathrm{am}\,u$, $\sin\mathrm{coam}\,u$, $\cos\mathrm{am}\,u$, $\cos\mathrm{coam}\,u$, $\Delta\mathrm{am}\,u$, $\Delta\mathrm{coam}\,u$, cet. ac generaliter *functiones trigonometricas amplitudinis*, in sequentibus *Functionum Ellipticarum* nomine insignire convenit."

The section concludes with the five formulæ

$$\sin\mathrm{coam}\,u = \frac{\cos\mathrm{am}\,u}{\Delta\mathrm{am}\,u},$$

$$\cos\mathrm{coam}\,u = \frac{k'\sin\mathrm{am}\,u}{\Delta\mathrm{am}\,u},$$

$$\Delta\mathrm{coam}\,u = \frac{k'}{\Delta\mathrm{am}\,u},$$

$$\tan \operatorname{coam} u = \frac{1}{k' \tan \operatorname{am} u},$$

$$\cot \operatorname{coam} u = \frac{k'}{\cot \operatorname{am} u}.$$

§ 2. Throughout the *Fundamenta Nova*, Jacobi makes as free use of the functions involving the coamplitude as of those involving the amplitude, and the systems of formulæ which he gives on pp. 70, 71, have reference to the eight functions

$\sin \operatorname{am} u$,	$\sin \operatorname{coam} u$,
$\cos \operatorname{am} u$,	$\cos \operatorname{coam} u$,
$\Delta \operatorname{am} u$,	$\Delta \operatorname{coam} u$,
$\tan \operatorname{am} u$,	$\tan \operatorname{coam} u$.

[On account of the length of Jacobi's functional notation, Gudermann* denoted these eight functions by

$\operatorname{sn} u$,	$\operatorname{snc} u$,
$\operatorname{cn} u$,	$\operatorname{cnc} u$,
$\operatorname{dn} u$,	$\operatorname{dnc} u$,
$\operatorname{tn} u$,	$\operatorname{tnc} u$,

and for the sake of brevity, it is generally convenient to employ this notation in preference to Jacobi's].

§ 3. It would appear from § 1, that Jacobi was led by the analogy between K and $\frac{1}{2}\pi$ to assume

$$\operatorname{am}(K - u) = \operatorname{coam} u,$$

since it thence follows that

$$\sin \operatorname{am}(K - u) = \sin \operatorname{coam} u,$$

$$\cos \operatorname{am}(K - u) = \cos \operatorname{coam} u,$$

and, therefore, when $k = 0$, $\sin \operatorname{coam} u$ and $\cos \operatorname{coam} u$ reduce to $\cos u$ and $\sin u$ respectively.

It is true that the use of the coamplitude function simplifies considerably the expression of many results, and in

* *Crelle's Journal*, vol. XVIII. (1838), pp. 12, 20.

some formulæ, such as for example

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn}u \operatorname{snc}v + \operatorname{sn}v \operatorname{snc}u}{1 + k^2 \operatorname{sn}u \operatorname{sn}v \operatorname{snc}u \operatorname{snc}v},$$

$$\operatorname{cn}(u+v) = \frac{\operatorname{cn}u \operatorname{cn}v - \operatorname{cnc}u \operatorname{cnc}v}{1 + \frac{k^2}{k'^2} \operatorname{cn}u \operatorname{cn}v \operatorname{cnc}u \operatorname{cnc}v},$$

the functions snc and cnc exhibit a close analogy to the cosine and sine; but on the other hand it seems undesirable that there should be a fundamental distinction in mode of expression between the formulæ relating to the real and imaginary periods, as there is no corresponding notation for $\operatorname{am}(iK' - u)$; and the fact that when $k = 0$, the real period $4K$ reduces to 2π, while $4iK'$ becomes infinite, does not afford sufficient reason for this difference of treatment in what is essentially the Theory of Double Periodicity.

It is also to be remarked, that the three fundamental functions $\operatorname{sn}u$, $\operatorname{cn}u$, $\operatorname{dn}u$, with their reciprocals and quotients form a system of twelve functions; but that Jacobi's functions mentioned in § 2 are only eight in number, and that if we include also the reciprocals of these eight, we obtain a system of sixteen functions, four of which must therefore be repetitions of four of the others.

The values of the sixteen functions expressed in terms of $\operatorname{sn}u$, $\operatorname{cn}u$, $\operatorname{dn}u$ are given by the equations

$$\sin\operatorname{am}u = \operatorname{sn}u, \qquad \sin\operatorname{coam}u = \frac{\operatorname{cn}u}{\operatorname{dn}u},$$

$$\cos\operatorname{am}u = \operatorname{cn}u, \qquad \cos\operatorname{coam}u = k' \frac{\operatorname{sn}u}{\operatorname{dn}u},$$

$$\Delta\operatorname{am}u = \operatorname{dn}u, \qquad \Delta\operatorname{coam}u = k' \frac{1}{\operatorname{dn}u},$$

$$\tan\operatorname{am}u = \frac{\operatorname{sn}u}{\operatorname{cn}u}, \qquad \tan\operatorname{coam}u = \frac{1}{k'} \frac{\operatorname{cn}u}{\operatorname{sn}u},$$

$$\frac{1}{\sin\operatorname{am}u} = \frac{1}{\operatorname{sn}u}, \qquad \frac{1}{\sin\operatorname{coam}u} = \frac{\operatorname{dn}u}{\operatorname{cn}u},$$

$$\frac{1}{\cos\operatorname{am}u} = \frac{1}{\operatorname{cn}u}, \qquad \frac{1}{\cos\operatorname{coam}u} = \frac{1}{k'} \frac{\operatorname{dn}u}{\operatorname{sn}u},$$

$$\frac{1}{\Delta\operatorname{am}u} = \frac{1}{\operatorname{dn}u}, \qquad \frac{1}{\Delta\operatorname{coam}u} = \frac{1}{k'} \operatorname{dn}u,$$

$$\frac{1}{\tan\operatorname{am}u} = \frac{\operatorname{cn}u}{\operatorname{sn}u}, \qquad \frac{1}{\tan\operatorname{coam}u} = k' \frac{\operatorname{sn}u}{\operatorname{cn}u},$$

the repetitions being due to the relations

$$\Delta \operatorname{am} u \quad \Delta \operatorname{coam} u = k',$$

$$\tan \operatorname{am} u \tan \operatorname{coam} u = \frac{1}{k'}.$$

Conversely, by expressing the twelve functions in terms of Jacobi's functions, we obtain the equations

$$\operatorname{sn} u = \quad \sin \operatorname{am} u,$$

$$\operatorname{cn} u = \quad \cos \operatorname{am} u,$$

$$\operatorname{dn} u = \quad \Delta \operatorname{am} u \quad = k' \frac{1}{\Delta \operatorname{coam} u},$$

$$\frac{1}{\operatorname{sn} u} = \quad \frac{1}{\sin \operatorname{am} u},$$

$$\frac{1}{\operatorname{cn} u} = \quad \frac{1}{\cos \operatorname{am} u},$$

$$\frac{1}{\operatorname{dn} u} = \quad \frac{1}{\Delta \operatorname{am} u} \quad = \frac{1}{k'} \Delta \operatorname{coam} u,$$

$$\frac{\operatorname{sn} u}{\operatorname{cn} u} = \quad \tan \operatorname{am} u \quad = \frac{1}{k'} \frac{1}{\tan \operatorname{coam} u},$$

$$\frac{\operatorname{sn} u}{\operatorname{dn} u} = \frac{1}{k'} \cos \operatorname{coam} u,$$

$$\frac{\operatorname{cn} u}{\operatorname{dn} u} = \quad \sin \operatorname{coam} u,$$

$$\frac{\operatorname{cn} u}{\operatorname{sn} u} = \quad \frac{1}{\tan \operatorname{am} u} \quad = k' \tan \operatorname{coam} u,$$

$$\frac{\operatorname{dn} u}{\operatorname{sn} u} = k' \frac{1}{\cos \operatorname{coam} u},$$

$$\frac{\operatorname{dn} u}{\operatorname{cn} u} = \quad \frac{1}{\sin \operatorname{coam} u}.$$

§ 4. It has been usual for writers on Elliptic Functions either to follow Jacobi in the use of the coamplitude function, or to select certain only of the functions (such as the sn, cn, dn, tn) for consideration; and, so far as I know, the

twelve functions have not been regarded hitherto in any treatment of the subject, as forming the members of a complete system of fundamental expressions in the Theory.

The object of the present series of papers is to give in some detail various systems of formulæ involving the twelve functions: several of these are merely extensions of well-known groups of results, but some of the others have not, I believe, been specially noticed previously.

The corresponding groups of formulæ involving the six primary circular functions (viz. the sine and cosine and their reciprocals and quotients) are also given, as the existence of many of the systems in Elliptic Functions is indicated by them, and as, on the other hand, it is interesting to remark the traces of the more extended theory which appear in these elementary results. The different formulæ of the same group show considerable diversity of form in the case of the circular functions; but the twelve elliptic functions are much more regular and uniform. This is what we should expect, as all the twelve elliptic functions, regarded algebraically, are fractional (*i.e.* involve both numerators and denominators), while of the trigonometrical functions, the sine and cosine are numerator-functions, the secant and cosecant denominator-functions, and the tangent and cotangent alone are fractional.

Notation, § 5.

§ 5. The twelve elliptic functions will generally be expressed in terms of $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$; that is to say, they will be written

$$\mathrm{sn}\,u,\quad \mathrm{cn}\,u,\quad \mathrm{dn}\,u,\quad \frac{1}{\mathrm{sn}\,u},\ \frac{1}{\mathrm{cn}\,u},\ \frac{1}{\mathrm{dn}\,u},$$

$$\frac{\mathrm{sn}\,u}{\mathrm{cn}\,u},\ \frac{\mathrm{sn}\,u}{\mathrm{dn}\,u},\ \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u},\ \frac{\mathrm{cn}\,u}{\mathrm{sn}\,u},\ \frac{\mathrm{dn}\,u}{\mathrm{sn}\,u},\ \frac{\mathrm{dn}\,u}{\mathrm{cn}\,u}.$$

It is useful however to have also a uniform notation, containing the same number of letters, for each of the twelve functions, and I have found it convenient to denote the quotient functions by pairs of letters consisting of the first letter of the numerator followed by the first letter of the denominator, and the reciprocals of $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$ by $\mathrm{ns}\,u$, $\mathrm{nc}\,u$, $\mathrm{nd}\,u$, the order of the letters being thus reversed, so that we have

$$\operatorname{sn} u = \operatorname{sn} u, \qquad \operatorname{sc} u = \frac{\operatorname{sn} u}{\operatorname{cn} u},$$

$$\operatorname{cn} u = \operatorname{cn} u, \qquad \operatorname{sd} u = \frac{\operatorname{sn} u}{\operatorname{dn} u},$$

$$\operatorname{dn} u = \operatorname{dn} u, \qquad \operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u},$$

$$\operatorname{ns} u = \frac{1}{\operatorname{sn} u}, \qquad \operatorname{cs} u = \frac{\operatorname{cn} u}{\operatorname{sn} u},$$

$$\operatorname{nc} u = \frac{1}{\operatorname{cn} u}, \qquad \operatorname{ds} u = \frac{\operatorname{dn} u}{\operatorname{sn} u},$$

$$\operatorname{nd} u = \frac{1}{\operatorname{dn} u}, \qquad \operatorname{dc} u = \frac{\operatorname{dn} u}{\operatorname{cn} u}.$$

Even and uneven functions, § 6.

§ 6. Of the six trigonometrical functions only two, the cosine and secant are even, the other four being uneven.

Of the twelve elliptic functions six are even and six are uneven; the uneven functions being those which involve sn, or which, in the notation of the last section, involve the letter s.

Change of the argument, §§ 7–10.

§ 7. We have in Trigonometry the following formulæ connecting the functions of the argument $\frac{1}{2}\pi + u$ with those of the argument u,*

$$\sin\left(\tfrac{1}{2}\pi + u\right) = \cos u, \qquad \cos\left(\tfrac{1}{2}\pi + u\right) = -\sin u,$$

$$\tan\left(\tfrac{1}{2}\pi + u\right) = -\cot u, \qquad \cot\left(\tfrac{1}{2}\pi + u\right) = -\tan u,$$

$$\sec\left(\tfrac{1}{2}\pi + u\right) = -\operatorname{cosec} u, \qquad \operatorname{cosec}\left(\tfrac{1}{2}\pi + u\right) = \sec u.$$

The six functions are thus divisible into six pairs, viz. $\sin u$ and $\cos u$, $\tan u$ and $\cot u$, $\sec u$ and $\operatorname{cosec} u$, which are such that if we select from them any triad, one being taken from each pair, the other three functions may be derived from this triad by a change in the argument. The number of such triads is eight, the total number of triads that can be formed from the six functions being twenty.

* It seems more natural in considering the effect of change of argument in trigonometrical functions to replace u by $\frac{1}{2}\pi - u$ than by $\frac{1}{2}\pi + u$; but the latter substitution has been preferred, as in Elliptic Functions it is somewhat more convenient to replace u by $K + u$, &c. than by $K - u$, &c.

§ 8. The changes produced by the substitution of $K+u$, $iK'+u$, and $K+iK'+u$ for u in the twelve elliptic functions are shown in the following table, in which the second, third, and fourth columns contain the expressions into which the functions in the first column are converted by the substitution of $K+u$, $iK'+u$, and $K+iK'+u$ for u.

	$K+u$	$iK'+u$	$K+iK'+u$
$\operatorname{sn} u$	$\dfrac{\operatorname{cn} u}{\operatorname{dn} u}$,	$\dfrac{1}{k}\dfrac{1}{\operatorname{sn} u}$,	$\dfrac{1}{k}\dfrac{\operatorname{dn} u}{\operatorname{cn} u}$
$\operatorname{cn} u$	$-k'\dfrac{\operatorname{sn} u}{\operatorname{dn} u}$,	$-\dfrac{i}{k}\dfrac{\operatorname{dn} u}{\operatorname{sn} u}$,	$-\dfrac{ik'}{k}\dfrac{1}{\operatorname{cn} u}$
$\operatorname{dn} u$	$k'\dfrac{1}{\operatorname{dn} u}$,	$-i\dfrac{\operatorname{cn} u}{\operatorname{sn} u}$,	$ik'\dfrac{\operatorname{sn} u}{\operatorname{cn} u}$
$\dfrac{1}{\operatorname{sn} u}$	$\dfrac{\operatorname{dn} u}{\operatorname{cn} u}$,	$k\operatorname{sn} u$,	$k\dfrac{\operatorname{cn} u}{\operatorname{dn} u}$
$\dfrac{1}{\operatorname{cn} u}$	$-\dfrac{1}{k'}\dfrac{\operatorname{dn} u}{\operatorname{sn} u}$,	$ik\dfrac{\operatorname{sn} u}{\operatorname{dn} u}$,	$\dfrac{ik}{k'}\operatorname{cn} u$
$\dfrac{1}{\operatorname{dn} u}$	$\dfrac{1}{k'}\operatorname{dn} u$,	$i\dfrac{\operatorname{sn} u}{\operatorname{cn} u}$,	$-\dfrac{i}{k'}\dfrac{\operatorname{cn} u}{\operatorname{sn} u}$
$\dfrac{\operatorname{sn} u}{\operatorname{cn} u}$	$-\dfrac{1}{k'}\dfrac{\operatorname{cn} u}{\operatorname{sn} u}$,	$i\dfrac{1}{\operatorname{dn} u}$,	$\dfrac{i}{k'}\operatorname{dn} u$
$\dfrac{\operatorname{sn} u}{\operatorname{dn} u}$	$\dfrac{1}{k'}\operatorname{cn} u$,	$\dfrac{i}{k}\dfrac{1}{\operatorname{cn} u}$,	$-\dfrac{i}{kk'}\dfrac{\operatorname{dn} u}{\operatorname{sn} u}$
$\dfrac{\operatorname{cn} u}{\operatorname{dn} u}$	$-\operatorname{sn} u$,	$\dfrac{1}{k}\dfrac{\operatorname{dn} u}{\operatorname{cn} u}$,	$-\dfrac{1}{k}\dfrac{1}{\operatorname{sn} u}$
$\dfrac{\operatorname{cn} u}{\operatorname{sn} u}$	$-k'\dfrac{\operatorname{sn} u}{\operatorname{cn} u}$,	$-i\operatorname{dn} u$,	$-ik'\dfrac{1}{\operatorname{dn} u}$
$\dfrac{\operatorname{dn} u}{\operatorname{sn} u}$	$k'\dfrac{1}{\operatorname{cn} u}$,	$-ik\operatorname{cn} u$,	$ikk'\dfrac{\operatorname{sn} u}{\operatorname{dn} u}$
$\dfrac{\operatorname{dn} u}{\operatorname{cn} u}$	$-\dfrac{1}{\operatorname{sn} u}$,	$k\dfrac{\operatorname{cn} u}{\operatorname{dn} u}$,	$-k\operatorname{sn} u$,

On account of its fundamental character it seems desirable to give this table also in the notation of § 5. The table thus becomes

	$K+u$	$iK'+u$	$K+iK'+u$
$\mathrm{sn}\,u$	$\mathrm{cd}\,u$,	$\frac{1}{k}\,\mathrm{ns}\,u$,	$\frac{1}{k}\,\mathrm{dc}\,u$
$\mathrm{cn}\,u$	$-k'\,\mathrm{sd}\,u$,	$-\frac{i}{k}\,\mathrm{ds}\,u$,	$-\frac{ik'}{k}\,\mathrm{nc}\,u$
$\mathrm{dn}\,u$	$k'\,\mathrm{nd}\,u$,	$-i\,\mathrm{cs}\,u$,	$ik'\,\mathrm{sc}\,u$
$\mathrm{ns}\,u$	$\mathrm{dc}\,u$,	$k\,\mathrm{sn}\,u$,	$k\,\mathrm{cd}\,u$
$\mathrm{nc}\,u$	$-\frac{1}{k'}\,\mathrm{ds}\,u$,	$ik\,\mathrm{sd}\,u$,	$\frac{ik}{k'}\,\mathrm{cn}\,u$
$\mathrm{nd}\,u$	$\frac{1}{k'}\,\mathrm{dn}\,u$,	$i\,\mathrm{sc}\,u$,	$-\frac{i}{k'}\,\mathrm{cs}\,u$
$\mathrm{sc}\,u$	$-\frac{1}{k'}\,\mathrm{cs}\,u$,	$i\,\mathrm{nd}\,u$,	$\frac{i}{k'}\,\mathrm{dn}\,u$
$\mathrm{sd}\,u$	$\frac{1}{k'}\,\mathrm{cn}\,u$,	$\frac{i}{k}\,\mathrm{nc}\,u$,	$-\frac{i}{kk'}\,\mathrm{ds}\,u$
$\mathrm{cd}\,u$	$-\mathrm{sn}\,u$,	$\frac{1}{k}\,\mathrm{dc}\,u$,	$-\frac{1}{k}\,\mathrm{ns}\,u$
$\mathrm{cs}\,u$	$-k'\,\mathrm{sc}\,u$,	$-i\,\mathrm{dn}\,u$,	$-ik'\,\mathrm{nd}\,u$
$\mathrm{ds}\,u$	$k'\,\mathrm{nc}\,u$,	$-ik\,\mathrm{cn}\,u$,	$ikk'\,\mathrm{sd}\,u$
$\mathrm{dc}\,u$	$-\mathrm{ns}\,u$,	$k\,\mathrm{cd}\,u$,	$-k\,\mathrm{sn}\,u$.

§ 9. The effect of the substitutions in transforming the functions is shown more clearly in the following tables, which differ from those in the last section only by the omission of the constant factors $\frac{1}{k}$, $-k'$, $-\frac{i}{k}$, &c. In the first table s, c, d, are written in place of $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$.

	$K+u$	$iK'+u$	$K+iK'+u$		$K+u$	$iK'+u$	$K+iK'+u$
s	$\frac{c}{d}$,	$\frac{1}{s}$,	$\frac{d}{c}$	$\frac{s}{c}$	$\frac{c}{s}$,	$\frac{1}{d}$,	d
c	$\frac{s}{d}$,	$\frac{d}{s}$,	$\frac{1}{c}$	$\frac{s}{d}$	c,	$\frac{1}{c}$,	$\frac{d}{s}$
d	$\frac{1}{d}$,	$\frac{c}{s}$,	$\frac{s}{c}$	$\frac{c}{d}$	s,	$\frac{d}{c}$,	$\frac{1}{s}$
$\frac{1}{s}$	$\frac{d}{c}$,	s,	$\frac{c}{d}$	$\frac{c}{s}$	$\frac{s}{c}$,	d,	$\frac{1}{d}$
$\frac{1}{c}$	$\frac{d}{s}$,	$\frac{s}{d}$,	c	$\frac{d}{s}$	$\frac{1}{c}$,	c,	$\frac{s}{d}$
$\frac{1}{d}$	d,	$\frac{s}{c}$,	$\frac{c}{s}$	$\frac{d}{c}$	$\frac{1}{s}$,	$\frac{c}{d}$,	s

The same table, expressed in the notation of § 5, is

	$K+u$	$iK'+u$	$K+iK'+u$
sn u	cd u,	ns u,	dc u
cn u	sd u,	ds u,	nc u
dn u	nd u,	cs u,	sc u
ns u	dc u,	sn u,	cd u
nc u	ds u,	sd u,	cn u
nd u	dn u,	sc u,	cs u
sc u	cs u,	nd u,	dn u
sd u	cn u,	nc u,	ds u
cd u	sn u,	dc u,	ns u
cs u	sc u,	dn u,	nd u
ds u	nc u,	cn u,	sd u
dc u	ns u,	cd u,	sn u

These changes may perhaps be rendered still more evident if we denote $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ by 1, 2, 3 and their reciprocals by 4, 5, 6; and if we denote $\frac{\operatorname{sn} u}{\operatorname{cn} u}$, $\frac{\operatorname{sn} u}{\operatorname{dn} u}$, $\frac{\operatorname{cn} u}{\operatorname{dn} u}$ by 7, 8, 9 and their reciprocals by 10, 11, 12. We then have the scheme

1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12;
9,	8,	6,	12,	11,	3,	10,	2,	1,	7,	5,	4;
4,	11,	10,	1,	8,	7,	6,	5,	12,	3,	2,	9;
12,	5,	7,	9,	2,	10,	3,	11,	4,	6,	8,	1;

the second, third, and fourth lines of which give the functions into which those in the first line are converted by the change of u into $K+u$, $iK'+u$, and $K+iK'+u$ respectively.

§ 10. The twelve functions are thus divisible into three groups of four each, viz.

$$\operatorname{sn} u, \quad \frac{1}{\operatorname{sn} u}, \quad \frac{\operatorname{dn} u}{\operatorname{cn} u}, \quad \frac{\operatorname{cn} u}{\operatorname{dn} u},$$

$$\operatorname{cn} u, \quad \frac{\operatorname{dn} u}{\operatorname{sn} u}, \quad \frac{1}{\operatorname{cn} u}, \quad \frac{\operatorname{sn} u}{\operatorname{dn} u},$$

$$\operatorname{dn} u, \quad \frac{\operatorname{cn} u}{\operatorname{sn} u}, \quad \frac{\operatorname{sn} u}{\operatorname{cn} u}, \quad \frac{1}{\operatorname{dn} u},$$

or, in the notation of § 5,

$$\begin{array}{llll} \operatorname{sn} u, & \operatorname{ns} u, & \operatorname{dc} u, & \operatorname{cd} u, \\ \operatorname{cn} u, & \operatorname{ds} u, & \operatorname{nc} u, & \operatorname{sd} u, \\ \operatorname{dn} u, & \operatorname{cs} u, & \operatorname{sc} u, & \operatorname{nd} u, \end{array}$$

which are such that if we select from them any triad, one being taken from each line, then the other nine functions may be derived from this triad by changes in the argument, that is, by the substitution of $K+u$, $iK'+u$, and $K+iK'+u$ for u. The number of such triads is 64, the total number of triads that can be formed from the twelve functions being 220.

Derivatives of the functions, §§ 11, 12.

§ 11. The derivatives of the six trigonometrical functions are given by the equations

$$\frac{d}{du}\sin u = \cos u, \qquad \frac{d}{du}\cos u = -\sin u,$$

$$\frac{d}{du}\tan u = \sec^2 u, \qquad \frac{d}{du}\cot u = -\operatorname{cosec}^2 u,$$

$$\frac{d}{du}\sec u = \sec u \tan u, \qquad \frac{d}{du}\operatorname{cosec} u = -\operatorname{cosec} u \cot u,$$

and it will be observed that they are very dissimilar in form.

§ 12. The derivatives of the twelve elliptic functions are given by the equations

$$\frac{d}{du}\operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u,$$

$$\frac{d}{du}\operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u,$$

$$\frac{d}{du}\operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u,$$

$$\frac{d}{du}\frac{1}{\operatorname{sn} u} = -\frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn}^2 u},$$

$$\frac{d}{du}\frac{1}{\operatorname{cn} u} = \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn}^2 u},$$

$$\frac{d}{du}\frac{1}{\operatorname{dn} u} = k^2\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn}^2 u},$$

$$\frac{d}{du}\frac{\operatorname{sn} u}{\operatorname{cn} u} = \frac{\operatorname{dn} u}{\operatorname{cn}^2 u},$$

$$\frac{d}{du}\frac{\operatorname{sn} u}{\operatorname{dn} u} = \frac{\operatorname{cn} u}{\operatorname{dn}^2 u},$$

$$\frac{d}{du}\frac{\operatorname{cn} u}{\operatorname{dn} u} = -k'^2\frac{\operatorname{sn} u}{\operatorname{dn}^2 u},$$

$$\frac{d}{du}\frac{\operatorname{cn} u}{\operatorname{sn} u} = -\frac{\operatorname{dn} u}{\operatorname{sn}^2 u},$$

$$\frac{d}{du}\frac{\operatorname{dn} u}{\operatorname{sn} u} = -\frac{\operatorname{cn} u}{\operatorname{sn}^2 u},$$

$$\frac{d}{du}\frac{\operatorname{dn} u}{\operatorname{cn} u} = k'^2\frac{\operatorname{sn} u}{\operatorname{cn}^2 u}.$$

Expressing these formulæ in the notation of § 5 and using an accent to denote differentiation, we have

$$\begin{aligned}
\mathrm{sn}'\,u &= \quad \mathrm{cn}\,u\,\mathrm{dn}\,u,\\
\mathrm{cn}'\,u &= -\quad \mathrm{sn}\,u\,\mathrm{dn}\,u,\\
\mathrm{dn}'\,u &= -k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u,\\
\mathrm{ns}'\,u &= -\quad \mathrm{cs}\,u\,\mathrm{ds}\,u,\\
\mathrm{nc}'\,u &= \quad \mathrm{sc}\,u\,\mathrm{dc}\,u,\\
\mathrm{nd}'\,u &= \quad k^2\,\mathrm{sd}\,u\,\mathrm{cd}\,u,\\
\mathrm{sc}'\,u &= \quad \mathrm{nc}\,u\,\mathrm{dc}\,u,\\
\mathrm{sd}'\,u &= \quad \mathrm{nd}\,u\,\mathrm{cd}\,u,\\
\mathrm{cd}'\,u &= -k'^2\,\mathrm{nd}\,u\,\mathrm{sd}\,u,\\
\mathrm{cs}'\,u &= -\quad \mathrm{ns}\,u\,\mathrm{ds}\,u,\\
\mathrm{ds}'\,u &= -\quad \mathrm{ns}\,u\,\mathrm{cs}\,u,\\
\mathrm{dc}'\,u &= \quad k'^2\,\mathrm{nc}\,u\,\mathrm{sc}\,u.
\end{aligned}$$

These formulæ may be divided into four groups, viz.

$$\left.\begin{aligned}
\mathrm{sn}'\,u &= \quad \mathrm{cn}\,u\,\mathrm{dn}\,u,\\
\mathrm{cn}'\,u &= -\quad \mathrm{sn}\,u\,\mathrm{dn}\,u,\\
\mathrm{dn}'\,u &= -k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u,
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\mathrm{ns}'\,u &= -\quad \mathrm{cs}\,u\,\mathrm{ds}\,u,\\
\mathrm{cs}'\,u &= -\quad \mathrm{ns}\,u\,\mathrm{ds}\,u,\\
\mathrm{ds}'\,u &= -\quad \mathrm{ns}\,u\,\mathrm{cs}\,u,
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\mathrm{nc}'\,u &= \quad \mathrm{sc}\,u\,\mathrm{dc}\,u,\\
\mathrm{sc}'\,u &= \quad \mathrm{nc}\,u\,\mathrm{dc}\,u,\\
\mathrm{dc}'\,u &= \quad k'^2\,\mathrm{nc}\,u\,\mathrm{sc}\,u,
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\mathrm{nd}'\,u &= \quad k^2\,\mathrm{sd}\,u\,\mathrm{cd}\,u,\\
\mathrm{sd}'\,u &= \quad \mathrm{nd}\,u\,\mathrm{cd}\,u,\\
\mathrm{cd}'\,u &= -k'^2\,\mathrm{nd}\,u\,\mathrm{sd}\,u.
\end{aligned}\right\}$$

The second, third, and fourth groups may be derived from the first group by the substitution of $iK'+u$, $K+iK'+u$, and $K+u$ for u respectively.

Integrals of the functions, §§ 13–15.

§ 13. In Trigonometry we have

$$\int \sin u\,du = -\cos u, \qquad \int \cos u\,du = \sin u,$$

$$\int \tan u\,du = -\log\cos u, \qquad \int \cot u\,du = \log\sin u,$$

$$\int \sec u\,du = \log\tan\left(\tfrac{1}{2}u+\tfrac{1}{4}\pi\right), \qquad \int \operatorname{cosec} u\,du = \log\tan\tfrac{1}{2}u,$$

four of the integrals being thus expressed as logarithmic functions.

§ 14. In Elliptic Functions the complete system of integrals may be expressed in the logarithmic form as follows:

$$\int \frac{1}{\operatorname{sn} u}\,du = -\log\frac{\operatorname{cn}\frac{1}{2}u\,\operatorname{dn}\frac{1}{2}u}{\operatorname{sn}\frac{1}{2}u},$$

$$\int \frac{\operatorname{dn} u}{\operatorname{sn} u}\,du = \log\frac{\operatorname{sn}\frac{1}{2}u\,\operatorname{dn}\frac{1}{2}u}{\operatorname{cn}\frac{1}{2}u},$$

$$\int \frac{\operatorname{cn} u}{\operatorname{sn} u}\,du = \log\frac{\operatorname{sn}\frac{1}{2}u\,\operatorname{cn}\frac{1}{2}u}{\operatorname{dn}\frac{1}{2}u},$$

$$\int \frac{\operatorname{dn} u}{\operatorname{cn} u}\,du = -\log\frac{\operatorname{cn}\left(\frac{1}{2}u+\frac{1}{2}K\right)\operatorname{dn}\left(\frac{1}{2}u+\frac{1}{2}K\right)}{\operatorname{sn}\left(\frac{1}{2}u+\frac{1}{2}K\right)},$$

$$\int \frac{1}{\operatorname{cn} u}\,du = \frac{1}{k'}\log\frac{\operatorname{sn}\left(\frac{1}{2}u+\frac{1}{2}K\right)\operatorname{dn}\left(\frac{1}{2}u+\frac{1}{2}K\right)}{\operatorname{cn}\left(\frac{1}{2}u+\frac{1}{2}K\right)},$$

$$\int \frac{\operatorname{sn} u}{\operatorname{cn} u}\,du = -\frac{1}{k'}\log\frac{\operatorname{sn}\left(\frac{1}{2}u+\frac{1}{2}K\right)\operatorname{cn}\left(\frac{1}{2}u+\frac{1}{2}K\right)}{\operatorname{dn}\left(\frac{1}{2}u+\frac{1}{2}K\right)},$$

$$\int \operatorname{sn} u\,du = -\frac{1}{k}\log\frac{\operatorname{cn}\left(\frac{1}{2}u+\frac{1}{2}iK'\right)\operatorname{dn}\left(\frac{1}{2}u+\frac{1}{2}iK'\right)}{\operatorname{sn}\left(\frac{1}{2}u+\frac{1}{2}iK'\right)},$$

$$\int \operatorname{cn} u\,du = \frac{i}{k}\log\frac{\operatorname{sn}\left(\frac{1}{2}u+\frac{1}{2}iK'\right)\operatorname{dn}\left(\frac{1}{2}u+\frac{1}{2}iK'\right)}{\operatorname{cn}\left(\frac{1}{2}u+\frac{1}{2}iK'\right)},$$

$$\int \operatorname{dn} u\,du = i\log\frac{\operatorname{sn}\left(\frac{1}{2}u+\frac{1}{2}iK'\right)\operatorname{cn}\left(\frac{1}{2}u+\frac{1}{2}iK'\right)}{\operatorname{dn}\left(\frac{1}{2}u+\frac{1}{2}iK'\right)},$$

$$\int \frac{\operatorname{cn} u}{\operatorname{dn} u}\,du = -\frac{1}{k}\log\frac{\operatorname{cn}\left(\frac{1}{2}u+\frac{1}{2}K+\frac{1}{2}iK'\right)\operatorname{dn}\left(\frac{1}{2}u+\frac{1}{2}K+\frac{1}{2}iK'\right)}{\operatorname{sn}\left(\frac{1}{2}u+\frac{1}{2}K+\frac{1}{2}iK'\right)},$$

$$\int \frac{\operatorname{sn} u}{\operatorname{dn} u}\,du = -\frac{i}{kk'}\log\frac{\operatorname{sn}\left(\frac{1}{2}u+\frac{1}{2}K+\frac{1}{2}iK'\right)\operatorname{dn}\left(\frac{1}{2}u+\frac{1}{2}K+\frac{1}{2}iK'\right)}{\operatorname{cn}\left(\frac{1}{2}u+\frac{1}{2}K+\frac{1}{2}iK'\right)},$$

$$\int \frac{1}{\operatorname{dn} u}\,du = \frac{i}{k'}\log\frac{\operatorname{sn}\left(\frac{1}{2}u+\frac{1}{2}K+\frac{1}{2}iK'\right)\operatorname{cn}\left(\frac{1}{2}u+\frac{1}{2}K+\frac{1}{2}iK'\right)}{\operatorname{dn}\left(\frac{1}{2}u+\frac{1}{2}K+\frac{1}{2}iK'\right)}.$$

§ 15. These integration formulæ may be expressed also in the form

$$\int \frac{1}{\operatorname{sn} u}\,du = -\log\sqrt{\left(\frac{\operatorname{dn} u+\operatorname{cn} u}{\operatorname{dn} u-\operatorname{cn} u}\right)},$$

$$\int \frac{\operatorname{dn} u}{\operatorname{sn} u}\,du = \log\sqrt{\left(\frac{1-\operatorname{cn} u}{1+\operatorname{cn} u}\right)},$$

$$\int \frac{\operatorname{cn} u}{\operatorname{sn} u}\,du = \log\sqrt{\left(\frac{1-\operatorname{dn} u}{1+\operatorname{dn} u}\right)},$$

$$\int \frac{\operatorname{dn} u}{\operatorname{cn} u}\,du = -\log\sqrt{\left(\frac{1-\operatorname{sn} u}{1+\operatorname{sn} u}\right)},$$

$$\int \frac{1}{\operatorname{cn} u}\,du = \frac{1}{k'}\log\sqrt{\left(\frac{\operatorname{dn} u+k'\operatorname{sn} u}{\operatorname{dn} u-k'\operatorname{sn} u}\right)},$$

$$\int \frac{\operatorname{sn} u}{\operatorname{cn} u}\,du = -\frac{1}{k'}\log\sqrt{\left(\frac{\operatorname{dn} u-k'}{\operatorname{dn} u+k'}\right)},$$

$$\int \operatorname{sn} n\,du = -\frac{1}{k}\log\sqrt{\left(\frac{\operatorname{dn} u+k\operatorname{cn} u}{\operatorname{dn} u-k\operatorname{cn} u}\right)},$$

$$\int \operatorname{cn} u\,du = \frac{i}{k}\log\sqrt{\left(\frac{k\operatorname{sn} u+i\operatorname{dn} u}{k\operatorname{sn} u-i\operatorname{dn} u}\right)},$$

$$\int \operatorname{dn} u\,du = i\log\sqrt{\left(\frac{\operatorname{sn} u+i\operatorname{cn} u}{\operatorname{sn} u-i\operatorname{cn} u}\right)},$$

$$\int \frac{\operatorname{cn} u}{\operatorname{dn} u}\,du = -\frac{1}{k}\log\sqrt{\left(\frac{1-k\operatorname{sn} u}{1+k\operatorname{sn} u}\right)},$$

$$\int \frac{\operatorname{sn} u}{\operatorname{dn} u}\,du = -\frac{i}{kk'}\log\sqrt{\left(\frac{k\operatorname{cn} u+ik'}{k\operatorname{cn} u-ik'}\right)},$$

$$\int \frac{1}{\operatorname{dn} u}\,du = \frac{i}{k'}\log\sqrt{\left(\frac{\operatorname{cn} u-ik'\operatorname{sn} u}{\operatorname{cn} u+ik'\operatorname{sn} u}\right)};$$

or in the form

$$\int \frac{1}{\operatorname{sn} u}\,du = -\log\frac{\operatorname{dn} u+\operatorname{cn} u}{\operatorname{sn} u},$$

$$\int \frac{\operatorname{dn} u}{\operatorname{sn} u}\,du = \log\frac{1-\operatorname{cn} u}{\operatorname{sn} u},$$

$$\int \frac{\operatorname{cn} u}{\operatorname{sn} u}\,du = \log\frac{1-\operatorname{dn} u}{\operatorname{sn} u},$$

$$\int \frac{\mathrm{dn}\,u}{\mathrm{cn}\,u}\,du = \quad -\log\frac{1-\mathrm{sn}\,u}{\mathrm{cn}\,u},$$

$$\int \frac{1}{\mathrm{cn}\,u}\,du = \quad \frac{1}{k'}\log\frac{\mathrm{dn}\,u+k'\,\mathrm{sn}\,u}{\mathrm{cn}\,u},$$

$$\int \frac{\mathrm{sn}\,u}{\mathrm{cn}\,u}\,du = -\frac{1}{k'}\log\frac{\mathrm{dn}\,u-k'}{\mathrm{cn}\,u},$$

$$\int \mathrm{sn}\,u\,du = \quad \frac{1}{k}\log(\mathrm{dn}\,u-k\,\mathrm{cn}\,u),$$

$$\int \mathrm{cn}\,u\,du = \quad \frac{i}{k}\log(\mathrm{dn}\,u-ik\,\mathrm{sn}\,u),$$

$$\int \mathrm{dn}\,u\,du = \quad i\log(\mathrm{cn}\,u-i\,\mathrm{sn}\,u),$$

$$\int \frac{\mathrm{cn}\,u}{\mathrm{dn}\,u}\,du = -\frac{1}{k}\log\frac{1-k\,\mathrm{sn}\,u}{\mathrm{dn}\,u},$$

$$\int \frac{\mathrm{sn}\,u}{\mathrm{dn}\,u}\,du = -\frac{i}{kk'}\log\frac{k\,\mathrm{cn}\,u+ik'}{\mathrm{dn}\,u},$$

$$\int \frac{1}{\mathrm{dn}\,u}\,dn = \quad \frac{i}{k'}\log\frac{\mathrm{cn}\,u-ik'\,\mathrm{sn}\,u}{\mathrm{dn}\,u};$$

the three forms of the values of the integrals being equal or differing only by a constant.

Only four of the values in either of the last two systems involve i, and these may be easily reduced to real forms by means of the equation

$$\mathrm{arc\,tan}\frac{B}{A} = \frac{1}{2i}\log\frac{A+iB}{A-iB},$$

thus giving the formulæ

$$\int \mathrm{cn}\,u\,du = \quad \frac{1}{k}\,\mathrm{arc\,tan}\left(\frac{k\,\mathrm{sn}\,u}{\mathrm{dn}\,u}\right),$$

$$\int \mathrm{dn}\,u\,du = \quad \mathrm{arc\,tan}\left(\frac{\mathrm{sn}\,u}{\mathrm{cn}\,u}\right) = \mathrm{am}\,u,$$

$$\int \frac{\mathrm{sn}\,u}{\mathrm{dn}\,u}\,du = \frac{1}{kk'}\,\mathrm{arc\,tan}\left(\frac{k'}{k\,\mathrm{cn}\,u}\right),$$

$$\int \frac{1}{\mathrm{dn}\,u}\,du = \quad \frac{1}{k'}\,\mathrm{arc\,tan}\left(\frac{k'\,\mathrm{sn}\,u}{\mathrm{cn}\,u}\right).$$

ON SOME FORMS OF COMPOUND DETERMINANTS.

By *R. F. Scott, M.A.*

STARTING with the determinants of order n

$$A = |\, a_{ik} \,|, \quad B = |\, b_{ik} \,|, \quad C = |\, c_{ik} \,|,$$

if we border the determinant A with a column from B and a row from C we obtain another system of $(n+1)^2$ elements

$$P_{ik} = \begin{vmatrix} a_{11}, \ldots a_{1n}, & b_{1k} \\ a_{21}, \ldots a_{2n}, & b_{2k} \\ \ldots\ldots\ldots\ldots & \\ a_{n1}, \ldots a_{nn}, & b_{nk} \\ c_{i1}, \ldots c_{in}, & 0 \end{vmatrix}.$$

We find the coefficient of b_{rk} in this determinant by erasing the rth row and $(n+1)$th column and giving to the determinant thus obtained the sign found by raising (-1) to the power $n+1+r$. In this new determinant we obtain the coefficient of c_{is} by erasing the sth column and nth row and giving to the determinant thus obtained the sign found by raising (-1) to the power $n+s$. The final determinant is the coefficient of $b_{rk}\, c_{is}$ in the determinant of order $(n+1)$; it is also obtained from the determinant A by erasing its rth row and sth column. To this minor in our case we must give the sign of the $(n+1+r+n+s)$th, *i.e.* of the $(r+s+1)$th power of (-1). But this same determinant, obtained by erasing the rth row and sth column of A if multiplied by the $(r+s)$th power of (-1) is the coefficient of a_{rs} in A. Therefore we have

$$P_{ik} = -\Sigma b_{rk}\, c_{is}\, A_{rs}.$$

Thus if

$$p_{ir} = c_{i1} A_{r1} + c_{i2} A_{r2} + \ldots + c_{in} A_{rn},$$

$$P_{ik} = -(b_{1k} p_{1r} + \ldots + b_{nk} p_{nr}).$$

Hence

$$\begin{aligned} |\, P_{ik} \,| &= (-1)^n \,|\, b_{ik} \,| \,.\, |\, p_{ik} \,| \\ &= (-1)^n \,|\, b_{ik} \,| \,.\, |\, c_{ik} \,| \,.\, |\, A_{ik} \,| \\ &= (-1)^n\, BCA^{n-1}. \end{aligned}$$

The result of bordering the determinant A with a row and column of the system reciprocal to c_{ik} and b_{ik} is perhaps worth noticing, viz. if

$$\pi_{ik} = \begin{vmatrix} a_{11}\ldots a_{1n}, & B_{1k} \\ \ldots\ldots\ldots\ldots & \\ a_{n1}\ldots a_{nn}, & B_{nk} \\ C_{i1}\ldots C_{in}, & 0 \end{vmatrix}.$$

We have in the same way

$$|\pi_{ik}| = (-1)^n |B_{ik}| \cdot |C_{ik}| \cdot |A_{ik}|$$
$$= (-1)^n (ABC)^{n-1}.$$

This result will also be obtained by bordering the determinant B with a row and column from the systems C_{ik}, A_{ik}, or by bordering the determinant C with a row and column from the systems A_{ik} and B_{ik}.

A similar theorem can be obtained for the case where we border A with m columns from B and m rows from C.

Let the minors of the determinants B and C of order m be denoted by the systems

$$\begin{matrix} \beta_{11}\ldots\beta_{1\mu}, & \gamma_{11}\ldots\gamma_{1\mu}, \\ \ldots\ldots\ldots, & \ldots\ldots\ldots, \\ \beta_{\mu_1}\ldots\beta_{\mu\mu}, & \gamma_{\mu_1}\ldots\gamma_{\mu\mu}, \end{matrix}$$

where $$\mu = n_m = \frac{n(n-1)\ldots(n-m+1)}{1.2\ldots m}.$$

It is assumed that the order of selection of the combinations of rows and columns is adhered to for both determinants. This order being taken also for the determinant A, and α'_{ik} being the complementary minor in A of what with the preceding notation would be called α_{ik}. Let P_{ik} now stand for the determinant obtained by bordering A with the ith combination of the rows of C and the kth combination of the columns of B; then, by an extension of Cauchy's theorem, we have

$$P_{ik} = (-1)^m \Sigma \beta_{rk} \gamma_{is} \alpha'_{rs}.$$

And, as before,

$$\begin{aligned} |\, P_{ik} \,| &= (-1)^{m\mu} \,|\, \beta_{ik} \,| \cdot |\, \gamma_{ik} \,| \cdot |\, \alpha'_{ik} \,| \\ &= (-1)^{m\mu} (BC)^{(n-1)_{m-1}} A^{(n-1)_m}. \end{aligned}$$

The extended form of Cauchy's theorem can be proved in the same manner as that theorem was proved in the early part of this note.

October 8th, 1881.

ON SOME ALTERNATING FUNCTIONS OF n VARIABLES.

By *R. F. Scott, M.A.*

1. I PROPOSE in the present paper to give simple demonstrations of some theorems already known, and also to give some new theorems of a similar character.

2. If we have n quantities $x_n, x_{n-1}, \ldots x_1$, then we have

$$(1 - x_1 x)(1 - x_2 x) \ldots (1 - x_n x) = 1 - p_1 x + p_2 x^2 - \ldots \pm p_n x^n \ldots (1),$$

where $p_1, p_2, \ldots p_n$ are the sums of products of the quantities $x_1, x_2, \ldots x_n$ taken respectively 1, 2, ...n at a time without repetition. If H_k stands for the sum of the homogeneous powers and products of $x_1, x_2, \ldots x_n$ taken k at a time with repetition, we have

$$\frac{1}{(1 - x_1 x)(1 - x_2 x) \ldots (1 - x_n x)} = 1 + H_1 x + H_2 x^2 + \ldots + H_k x^k + \ldots (2).$$

If now we write

$$s_k = \frac{1}{k} (x_n^{\,k} + x_{n-1}^{\,k} + \ldots + x_1^{\,k}),$$

we have, taking logarithms of equation (1),

$$-s_1 x - s_2 x^2 - \ldots - s_k x^k - \ldots = \log (1 - p_1 x + p_2 x^2 - \ldots \pm p_n x^n).$$

On differentiating, we obtain

$$-s_1 - 2 s_2 x - \ldots - k s_k x^{k-1} - \ldots = \frac{-p_1 + 2 p_2 x - \ldots \pm n p_n x^n}{1 - p_1 x + p_2 x^2 - \ldots \pm p_n x^n}.$$

Clearing this equation of fractions and equating coefficients of the powers of x, we obtain

$$\begin{aligned} p_1 &= s_1, \\ s_1 p_1 - 2p_2 &= 2s_2, \\ 2s_2 p_1 - s_1 p_2 + 3p_3 &= 3s_3, \\ &\cdots\cdots\cdots\cdots, \\ (k-1)s_{k-1} p_1 - (k-2)s_{k-2} p_2 + \ldots + kp_k &= ks_k, \end{aligned}$$

whence

$$k!\, p_k = \begin{vmatrix} s_1, & 1, & 0, & 0, & \ldots & 0 \\ 2s_2, & s_1, & 2, & 0, & \ldots & 0 \\ 3s_3, & 2s_2, & s_1, & 3, & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (k-1)s_{k-1}, & (k-2)s_{k-2}, & \cdots & \cdots & \cdots & (k-1) \\ ks_k, & (k-1)s_{k-1}, & \cdots & \cdots & \cdots & s_1 \end{vmatrix}.$$

Hence we see that on expanding the determinant on the right, and dividing both sides by $k!$, we have

$$p_k = (-1)^{k-1} s_k + \ldots,$$

the terms after the first being powers and products of $s_{k-1}, \ldots s_1$.

Similarly, taking the logarithm of (2), we have

$$s_1 x + s_2 x^2 + \ldots = \log(1 + H_1 x + H_2 x^2 + \ldots),$$

whence, differentiating,

$$s_1 + 2s_2 x^2 + \ldots = \frac{H_1 + 2H_2 x + 3H_3 x^2 + \ldots}{1 + H_1 x + H_2 x^2 + \ldots};$$

from which, by clearing of fractions and equating the coefficients of the powers of x, we obtain, in like manner,

$$k!\, H_k = \begin{vmatrix} s_1, & 1, & 0, & \ldots & 0 \\ -2s_2, & s_1, & -2, & \ldots & 0 \\ -3s_3, & 2s_2, & s_1, & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -(k-1)s_{k-1}, & (k-2)s_{k-2}, & \cdots & \cdots & -(k-1) \\ -ks_k, & (k-1)s_{k-1}, & \cdots & \cdots & s_1 \end{vmatrix}.$$

And hence, as before,

$$H_k = s_k + \ldots,$$

the terms after the first being powers and products of $H_{k-1}, \ldots H_1$.

These functions H_k are called the functions *aleph* by continental mathematicians; the name was given by Wronski, who denoted them by that Hebrew letter.

3. Regarding the functions s_n, s_{n-1}, $\ldots s_1$ as functions of the n variables x_n, x_{n-1}, $\ldots x_1$, we can find their Jacobian, and we have in fact

$$\frac{d(s_n, s_{n-1}, \ldots s_1)}{d(x_n, x_{n-1}, \ldots x_1)} = \begin{vmatrix} x_n^{n-1}, & x_{n-1}^{n-1}, & \ldots x_1^{n-1} \\ x_n^{n-2}, & x_{n-1}^{n-2}, & \ldots x_1^{n-2} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_n, & x_{n-1}, & \ldots x_1 \\ 1, & 1, & \ldots 1 \end{vmatrix}.$$

The determinant on the right is equal to the product of the differences

$$\begin{aligned} &(x_n - x_{n-1})(x_n - x_{n-2})\ldots(x_n - x_1), \\ &(x_{n-1} - x_{n-2})\ldots(x_{n-1} - x_1), \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ &(x_2 - x_1); \end{aligned}$$

or, with Prof. Sylvester's notation, $\zeta^{\frac{1}{2}}(x_n, x_{n-1}, \ldots x_1)$.

Hence

$$\frac{d(s_n, s_{n-1}, \ldots s_1)}{d(x_n, x_{n-1}, \ldots x_1)} = \zeta^{\frac{1}{2}}(x_n, x_{n-1}, \ldots x_1).$$

4. Now take any n of these sums of like powers, say s_p, s_q, $\ldots s_t$, where p, q, $\ldots t$ are any n integers arranged in descending order of magnitude. Then

$$\frac{d(s_p, s_q, \ldots s_t)}{d(x_n, x_{n-1}, \ldots x_1)} = \begin{vmatrix} x_n^{p-1}, & x_{n-1}^{p-1}, & \ldots x_1^{p-1} \\ x_n^{q-1}, & x_{n-1}^{q-1}, & \ldots x_1^{q-1} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ x_n^{t-1}, & x_{n-1}^{t-1}, & \ldots x_1^{t-1} \end{vmatrix}.$$

The determinant on the right has been considered by Jacobi (*De functionibus alternantibus earumque divisione per productum e differentiis elementorum conflatum. Crelle* XXII. p. 360 [1841]), and also, independently, and by a different method, by Sig. N. Trudi (*Intorno ad un determinante più generale di quello delle radici delle equazioni ed alle funzioni omogenee complete di questi radici*, "Giornale di Matematiche" II. p. 152. Also printed in the "Rendiconto dell' Accademia delle Scienze Fisiche e Matematiche" III. p. 121 [1864]). By these mathematicians it is shown that the value of the determinant is

$$\begin{vmatrix} H_{p-1}, & H_{p-2}, & \ldots H_{p-n} \\ H_{q-1}, & H_{q-2}, & \ldots H_{q-n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ H_{t-1}, & H_{t-2}, & \ldots H_{t-n} \end{vmatrix} \zeta^{\frac{1}{2}}(x_n, x_{n-1}, \ldots x_1),$$

which is therefore the value of

$$\frac{d(s_p, s_q, \ldots s_t)}{d(x_n, x_{n-1}, \ldots x_1)}.$$

5. Let us now consider the Jacobian

$$\frac{d(p_n, p_{n-1}, \ldots p_1)}{d(x_n, x_{n-1}, \ldots x_1)}.$$

By a known theorem this is equal to

$$\frac{d(p_n, p_{n-1}, \ldots p_1)}{d(s_n, s_{n-1}, \ldots s_1)} \cdot \frac{d(s_n, s_{n-1}, \ldots s_1)}{d(x_n, x_{n-1}, \ldots x_1)}.$$

In the first of these latter Jacobians all the elements below the principal diagonal of the determinant vanish, while the elements in the diagonal by § 2 are all equal to unity. The second Jacobian was evaluated in § 3. Hence

$$\frac{d(p_n, p_{n-1}, \ldots p_1)}{d(x_n, x_{n-1}, \ldots x_1)} = (-1)^{\frac{1}{2}n(n-1)} \zeta^{\frac{1}{2}}(x_n, x_{n-1}, \ldots x_1).$$

This theorem was proposed as a question in the "Educational Times" by Mr. F. A. Tarleton, "Educational Times" Reprint IX. p. 69 [1868].

6. Again, let us consider the Jacobian

$$\frac{d(H_n, H_{n-1}, \ldots H_1)}{d(x_n, x_{n-1}, \ldots x_1)}.$$

Transforming this to the product

$$\frac{d(H_n, H_{n-1}, \ldots H_1)}{d(s_n, s_{n-1}, \ldots s_1)} \cdot \frac{d(s_n, s_{n-1}, \ldots s_1)}{d(x_n, x_{n-1}, \ldots x_1)},$$

as before, the first Jacobian reduces to unity, and hence

$$\frac{d(H_n, H_{n-1}, \ldots H_1)}{d(x_n, x_{n-1}, \ldots x_1)} = \zeta^{\frac{1}{2}}(x_n, x_{n-1}, \ldots x_1).$$

This theorem is due to Sig. L. Crocchi (*Sopra le Funzioni Aleph ed il determinante di Cauchy*, "Giornale" XVII. p. 218 [1879]).

7. The same transformation gives us readily some theorems which, I believe, have not hitherto been noticed.

If $p, q, \ldots t$ be n integers in descending order of magnitude, we have

$$\frac{d(H_p, H_q, \ldots H_t)}{d(x_n, x_{n-1}, \ldots x_1)} = \frac{d(H_p \ldots H_t)}{d(s_p \ldots s_t)} \cdot \frac{d(s_p \ldots s_t)}{d(x_n \ldots x_1)}.$$

In the first determinant on the right, since $p, q, \ldots t$ are in descending order of magnitude, all the elements below the principal diagonal vanish, while those in the diagonal are all equal to unity. The second determinant has been evaluated in § 4. Hence

$$\frac{d(H_p, H_q, \ldots H_t)}{d(x_n, x_{n-1}, \ldots x_1)} = \begin{vmatrix} H_{p-1}, & H_{p-2}, & \ldots H_{p-n} \\ H_{q-1}, & H_{q-2}, & \ldots H_{q-n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ H_{t-1}, & H_{t-2}, & \ldots H_{t-n} \end{vmatrix} \zeta^{\frac{1}{2}}(x_n \ldots x_1),$$

8. Similarly,

$$\begin{aligned} \frac{d(H_p, H_q, \ldots H_t)}{d(p_n, p_{n-1}, \ldots p_1)} &= \frac{d(H_p, H_q, \ldots H_t)}{d(s_p, s_q, \ldots s_t)} \cdot \frac{d(s_p, \ldots s_t)}{d(p_n, \ldots p_1)} \\ &= \frac{d(s_p, s_q, \ldots s_t)}{d(x_n, x_{n-1} \ldots x_1)} \cdot \frac{d(x_n, x_{n-1}, \ldots x_1)}{d(p_n, p_{n-1}, \ldots p_1)} \\ &= (-1)^{\frac{1}{2}n(n-1)} \begin{vmatrix} H_{p-1}, & H_{p-2}, & \ldots H_{p-n} \\ H_{q-1}, & H_{q-2}, & \ldots H_{q-n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ H_{t-1}, & H_{t-2}, & \ldots H_{t-n} \end{vmatrix}, \end{aligned}$$

by aid of §§ 4, 5. The Jacobians

$$\frac{d(s_p, s_q, \ldots s_t)}{d(s_n, s_{n-1}, \ldots s_1)} \quad \text{and} \quad \frac{d(H_p, H_q, \ldots H_t)}{d(H_n, H_{n-1}, \ldots H_1)}$$

are equal to the same determinant.

If we choose $q, \ldots t = n-2, \ldots 1$ respectively, the determinant reduces to $(-1)^{\frac{1}{2}\{n(n-1)\}} H_{p-n}$.

9. If we have

$$\frac{1}{(1 - x_1 x . 1 - x_2 x \ldots 1 - x_n x)^u} = 1 + W_1 x + W_2 x^2 + \ldots,$$

we have

$$1 + W_1 x + W_2 x^2 + \ldots = (1 + H_1 x + H_2 x^2 + \ldots)^u.$$

Hence, on expanding the multinomial on the right, we obtain

$$W_k = uH_k + \ldots,$$

the remaining terms being powers and products of $H_{k-1}, \ldots H_1$.

Hence

$$W_k = us_k + \ldots,$$

the terms after the first being powers and products of $s_{k-1}, \ldots s_1$.

Hence, by reasoning similar to the preceding, we see that

$$\frac{d(W_n, W_{n-1}, \ldots W_1)}{d(x_n, x_{n-1}, \ldots x_1)} = u^n \zeta^2 (x_n, x_{n-1}, \ldots x_1).$$

Crocchi (*l. c.*) gives some transformations of this Jacobian, but does not effect its evaluation.

Similarly, we have

$$\frac{d(W_p, W_q, \ldots W_t)}{d(x_n, x_{n-1}, \ldots x_1)} = u^n \begin{vmatrix} H_{p-1}, & H_{p-2}, & \ldots H_{p-n} \\ H_{q-1}, & H_{q-2}, & \ldots H_{q-n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ H_{t-1}, & H_{t-2}, & \ldots H_{t-n} \end{vmatrix} \zeta^2 (x_n, x_{n-1}, \ldots x_1).$$

10. It is clear that the success of the preceding transformation depends on the fact that the functions p_k, H_k, W_k are linear in s_k, and that the same artifice will succeed for any series of functions, such that the function of rank k depends only on $s_k, s_{k-1}, \ldots s_1$, and is linear in s_k.

For example, since

$$D_m = \Sigma (x_p - x_q)^m$$
$$= nms_m - m(m-1)s_1 s_{m-1} + m(m-1)(m-2)s_2 s_{m-2} - \ldots,$$

we see that we can write down theorems similar to those already given; for example,

$$\frac{d(D_n, D_{n-1}, \ldots D_1)}{d(x_n, x_{n-1}, \ldots x_1)} = n^n n! \zeta^2 (x_n, x_{n-1}, \ldots x_1).$$

Or, again, for symmetrical functions, such as

$$\Sigma x_i^p x_j^q x_k^r = 2(p+q+r) s_{p+q+r} - \&c.,$$

we have theorems of the same kind differing only in the constant multipliers.

NOTES ON AREAL COORDINATES.

By *Harry Hart, M.A.*

LET the equation of a conic in areal coordinates be

$$\phi(xyz) \equiv ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0.$$

Let D, B stand respectively for the discriminant and the discriminant bordered with 1's, so that, writing 2α, 2β, 2γ for $v+w-2u'$, $w+u-2v'$, $u+v-2w'$, we have

$$B = \begin{vmatrix} u, & w', & v', & 1 \\ w', & v, & u', & 1 \\ v', & u', & w', & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix} = \alpha^2 + \beta^2 + \gamma^2 - 2\beta\gamma - 2\gamma\alpha - 2\alpha\beta.$$

1. The equation giving the lengths of the semi-axes of the conic may be written

$$\pm\left(2\alpha r^2 - \frac{D}{B}a^2\right)^{\frac{1}{2}} \pm \left(2\beta r^2 - \frac{D}{B}b^2\right)^{\frac{1}{2}} \pm \left(2\gamma r^2 - \frac{D}{B}c^2\right)^{\frac{1}{2}} = 0.$$

2. If θ be the angle between the asymptotes

$$\tan^2\theta = \frac{B}{(\alpha\cot A + \beta\cot B + \gamma\cot C)^2},$$

hence it is easily seen that the angle between the asymptotes is equal to that at which the circle circumscribing the triangle of reference is cut by the line

$$\alpha x + \beta y + \gamma z = 0.$$

3. The general equation represents a circle if

$$2\alpha r^2 - \frac{D}{B}a^2 = 2\beta r^2 - \frac{D}{B}b^2 = 2\gamma r^2 - \frac{D}{B}c^2 = 0,$$

which is equivalent to

$$\frac{2\alpha}{a^2} = \frac{2\beta}{b^2} = \frac{2\gamma}{c^2} = -\left\{\frac{D}{4\Delta^2 r^2}\right\}^{\frac{1}{2}},$$

Δ being the area of the triangle of reference.

4. The equation of the two tangents drawn from the point f, g, h is

$$\begin{vmatrix} gz-hy, & hx-fz, & fy-gx \\ f, & g, & b \\ u+w'+v', & w'+v+u', & v'+u'+w \end{vmatrix}^2 = -D.\phi(fgh) \begin{vmatrix} x, & y, & z \\ f, & g, & h \\ 1, & 1, & 1 \end{vmatrix}^2,$$

so that the curve is concave or convex to the point f, g, h according as $\phi(f, g, h)$ has or has not the same sign as the discriminant D.

The last equation represents the asymptotes if f, g, h are the coordinates of the centre; in this case the equation may be written also

$$\{B(u+w'+v')+3D\}x+\{B(w'+v+u')+3D\}y + \{B(v'+u'+w)+3D\}z = \pm B^{\frac{1}{2}} \begin{vmatrix} y-z, & z-x, & x-y, & 0 \\ u\ , & w'\ , & v'\ , & 1 \\ w'\ , & v\ , & u'\ , & 1 \\ v'\ , & u'\ , & w\ , & 1 \end{vmatrix}.$$

ON THE RESOLUTION OF A CERTAIN DETERMINANT INTO QUADRATIC FACTORS.

By *Thomas Muir, M.A., F.R.S.E.*

1. If we take two circulants, of the 5th order say, $C(a_1 \ldots, a_5)$, $C(b_1 \ldots, b_5)$, and write the first in the form

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_1 \end{vmatrix},$$

viz. so that the principal diagonal has all its elements alike, and the other in the form

$$\begin{vmatrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ b_2 & b_3 & b_4 & b_5 & b_1 \\ b_3 & b_4 & b_5 & b_1 & b_2 \\ b_4 & b_5 & b_1 & b_2 & b_3 \\ b_5 & b_1 & b_2 & b_3 & b_4 \end{vmatrix},$$

viz. so that the diagonal which has all its elements alike is now the secondary diagonal, we see that if every element

in the first be increased by the corresponding element in the second, there results a determinant

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 & a_4+b_4 & a_5+b_5 \\ a_5+b_2 & a_1+b_3 & a_2+b_4 & a_3+b_5 & a_4+b_1 \\ a_4+b_3 & a_5+b_4 & a_1+b_5 & a_2+b_1 & a_3+b_2 \\ a_3+b_4 & a_4+b_5 & a_5+b_1 & a_1+b_2 & a_2+b_3 \\ a_2+b_5 & a_3+b_1 & a_4+b_2 & a_5+b_3 & a_1+b_4 \end{vmatrix},$$

in which all the elements are different. This determinant is in a high degree noteworthy, from being resolvable into quadratic factors of a very peculiar form.

It is evidently the *general* eliminant of the set of equations

$$\left.\begin{aligned} (a_1+b_1)x_1+(a_2+b_2)x_2+(a_3+b_3)x_3+(a_4+b_4)x_4+(a_5+b_5)x_5=0, \\ (a_5+b_2)x_1+(a_1+b_3)x_2+(a_2+b_4)x_3+(a_3+b_5)x_4+(a_4+b_1)x_5=0, \\ (a_4+b_3)x_1+(a_5+b_4)x_2+(a_1+b_5)x_3+(a_2+b_1)x_4+(a_3+b_2)x_5=0, \\ (a_3+b_4)x_1+(a_4+b_5)x_2+(a_5+b_1)x_3+(a_1+b_2)x_4+(a_2+b_3)x_5=0, \\ (a_2+b_5)x_1+(a_3+b_1)x_2+(a_4+b_2)x_3+(a_5+b_3)x_4+(a_1+b_4)x_5=0, \end{aligned}\right\}$$

and, as we can obtain by a different process a *particular* eliminant of the system, the composition of the general eliminant is thus made evident. Denoting one of the imaginary fifth roots of unity by ω and multiplying both members of the 1st, 2nd, 3rd,... equations by ω^0, ω^1, ω^2... respectively, we have, by addition,

$$\left.\begin{aligned} &a_1(x_1+\omega x_2+\omega^2x_3+\omega^3x_4+\omega^4x_5)+b_1(x_1+\omega x_5+\omega^2x_4+\omega^3x_3+\omega^4x_2) \\ &+a_2(x_2+\omega x_3+\omega^2x_4+\omega^3x_5+\omega^4x_1)+b_2(x_2+\omega x_1+\omega^2x_5+\omega^3x_4+\omega^4x_3) \\ &+a_3(x_3+\omega x_4+\omega^2x_5+\omega^3x_1+\omega^4x_2)+b_3(x_3+\omega x_2+\omega^2x_1+\omega^3x_5+\omega^4x_4) \\ &+a_4(x_4+\omega x_5+\omega^2x_1+\omega^3x_2+\omega^4x_3)+b_4(x_4+\omega x_3+\omega^2x_2+\omega^3x_1+\omega^4x_5) \\ &+a_5(x_5+\omega x_1+\omega^2x_2+\omega^3x_3+\omega^4x_4)+b_5(x_5+\omega x_4+\omega^2x_3+\omega^3x_2+\omega^4x_1) \end{aligned}\right\}$$

$$=0;$$

and, therefore,

$$(a_1+\omega^{-1}a_2+\omega^{-2}a_3+\omega^{-3}a_4+\omega^{-4}a_5)(x_1+\omega x_2+\omega^2x_3+\omega^3x_4+\omega^4x_5)$$
$$+(b_1+\omega\ b_2+\omega^2\ b_3+\omega^3\ b_4+\omega^4\ b_5)(x_1+\omega x_5+\omega^2x_4+\omega^3x_3+\omega^4x_2)=0,$$

or

$$\frac{a_1+\omega^{-1}a_2+\omega^{-2}a_3+\omega^{-3}a_4+\omega^{-4}a_5}{b_1+\omega\ b_2+\omega^2\ b_3+\omega^3\ b_4+\omega^4\ b_5}=-\frac{x_1+\omega x_5+\omega^2x_4+\omega^3x_3+\omega^4x_2}{x_1+\omega x_2+\omega^2x_3+\omega^3x_4+\omega^4x_5}$$

....(1).

From this, by writing ω^{-1} for ω, we have

$$\frac{a_1+\omega\ a_2+\omega^2\ a_3+\omega^3\ a_4+\omega^4\ a_5}{b_1+\omega^{-1}b_2+\omega^{-2}b_3+\omega^{-3}b_4+\omega^{-4}b_5} = -\frac{x_1+\omega^{-1}x_5+\omega^{-2}x_4+\omega^{-3}x_3+\omega^{-4}x_2}{x_1+\omega^{-1}x_2+\omega^{-2}x_3+\omega^{-3}x_4+\omega^{-4}x_5}$$

and, therefore,

$$= -\frac{x_1+\omega^4x_5+\omega^3x_4+\omega^2x_3+\omega x_2}{x_1+\omega^4x_2+\omega^3x_3+\omega^2x_4+\omega x_5} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

if numerator and denominator be multiplied by ω^5 or 1.

Now the right-hand members of (1) and (2) are reciprocals, and, therefore, by multiplication, there results

$$\frac{(a_1+\omega^{-1}a_2+\omega^{-2}a_3+\omega^{-3}a_4+\omega^{-4}a_5)(a_1+\omega\ a_2+\omega^2\ a_3+\omega^3\ a_4+\omega^4\ a_5)}{(b_1+\omega\ b_2+\omega^2\ b_3+\omega^3\ b_4+\omega^4\ b_5)(b_1+\omega^{-1}b_2+\omega^{-2}b_3+\omega^{-3}b_4+\omega^{-4}b_5)} = 1,$$

or

$$(a_1+\omega a_2+\omega^2a_3+\omega^3a_4+\omega^4a_5)\,(a_1+\omega^{-1}a_2+\omega^{-2}a_3+\omega^{-3}a_4+\omega^{-4}a_5)$$
$$-(b_1+\omega b_2+\omega^2b_3+\omega^3b_4+\omega^4b_5)\,(b_1+\omega^{-1}b_2+\omega^{-2}b_3+\omega^{-3}b_4+\omega^{-4}b_5)=0.$$

The left-hand member of this equation being thus a particular eliminant of the set of equations is a factor of the determinant with which we started, and we have the general theorem:

The determinant whose every element is the sum of the corresponding elements of the circulants

$$C(a_1\ldots,\ a_n),\ \ (-1)^{\frac{1}{2}n(n-1)}C(b_1\ldots,\ b_n)$$

is divisible by

$$(a_1+\omega a_2+\omega^2a_3+\ldots+\omega^{n-1}a_n)\,(a_1+\omega^{-1}a_2+\omega^{-2}a_3+\ldots+\omega^{-n+1}a_n)$$
$$-(b_1+\omega b_2+\omega^2b_3+\ldots+\omega^{n-1}b_n)\,(b_1+\omega^{-1}b_2+\omega^{-2}b_3+\ldots+\omega^{-n+1}b_n),$$

ω being one of the imaginary n^{th} roots of unity.

If the determinant be of odd order there is evidently an additional linear factor, viz.

$$a_1+a_2+\ldots+a_n+b_1+b_2+\ldots+b_n;$$

if it be of even order there are two such, viz.

$$a_1+a_2+\ldots+a_n+b_1+b_2+\ldots+b_n,$$
$$a_1-a_2+\ldots-a_n+b_1-b_2+\ldots-b_n.$$

Putting $b_n=-x$ and $b_1=b_2=\ldots=b_{n-1}=0$ we have Mr. Glaisher's theorem, given in the *Quart. Journ. of Math.* XV. pp. 347–350. The mode of proof here followed is perfectly similar to that which is there used.

2. The quadratic factors may be put in a form free from imaginaries by performing the multiplications indicated in

them. Thus returning to the particular case of the fifth order, we have

$$(a_1+\omega a_2+\omega^2 a_3+\omega^3 a_4+\omega^4 a_5)(a_1+\omega^{-1}a_2+\omega^{-2}a_3+\omega^{-2}a_4+\omega^{-4}a_5)$$
$$= a_1^2+a_2^2+a_3^2+a_4^2+a_5^2$$
$$+ a_1a_2(\omega+\omega^{-1})+a_1a_3(\omega^2+\omega^{-2})+a_1a_4(\omega^3+\omega^{-3})+a_1a_5(\omega^4+\omega^{-4})$$
$$+ a_2a_3(\omega+\omega^{-1})+a_2a_4(\omega^2+\omega^{-2})+a_2a_5(\omega^3+\omega^{-3})$$
$$+ a_3a_4(\omega+\omega^{-1})+a_3a_5(\omega^2+\omega^{-2})$$
$$+ a_4a_5(\omega+\omega^{-1}).$$

But $\omega^4+\omega^{-4}=\omega+\omega^{-1}=2\cos\frac{1}{5}(2s\pi)$,

and $\omega^3+\omega^{-3}=\omega^2+\omega^{-2}=2\cos\frac{1}{5}(4s\pi)$,

s being any integer greater than 0 and less than 5.

Hence, if we use the notation of *cyclic sums*, viz. put

$\Sigma^0 a_1a_2$ for $a_1a_2+a_2a_3+a_3a_4+a_4a_5+a_5a_1$,

and $\Sigma^0 a_1a_3$ for $a_1a_3+a_2a_4+a_3a_5+a_4a_1+a_5a_2$,

we have the above product equal to

$$\Sigma^0 a_1^2+2\cos\tfrac{1}{5}(2s\pi)\,\Sigma^0 a_1a_2+2\cos\tfrac{1}{5}(4s\pi)\,\Sigma^0 a_1a_3;$$

and thus the factors of the determinant are

$$\Sigma^0 a_1+\Sigma^0 b_1,$$
$$\Sigma^0 a_1^2+2\cos\tfrac{2}{5}\pi\;\Sigma^0 a_1a_2-2\cos\tfrac{1}{5}\pi\,\Sigma^0 a_1a_3-\Sigma^0 b_1^2-\Sigma^0 b_1^2-2\cos\tfrac{2}{5}\pi\,\Sigma^0 b_1b_2+2\cos\tfrac{1}{5}\pi\,\Sigma^0 b_1b_3,$$
$$\Sigma^0 a_1^2-2\cos\tfrac{1}{5}\pi\;\Sigma^0 a_1a_2-2\cos\tfrac{2}{5}\pi\,\Sigma^0 a_1a_3-\Sigma^0 b_1^2-\Sigma^0 b_1^2+2\cos\tfrac{1}{5}\pi\,\Sigma^0 b_1b_2-2\cos\tfrac{1}{5}\pi\,\Sigma^0 b_1b_3.$$

Generally, if the determinant be of even order, the typical quadratic factor is

$$\left(\Sigma^0 a_1^2+2\cos\frac{2s\pi}{n}\,\Sigma^0 a_1a_2+2\cos\frac{2.2s\pi}{n}\,\Sigma^0 a_1a_3+\ldots+2\cos\frac{\frac{1}{2}n.2s\pi}{n}\,\Sigma^0 a_1a_{\frac{1}{2}n+1}\right)$$
$$-\left(\Sigma^0 b_1^2+2\cos\frac{2s\pi}{n}\,\Sigma^0 b_1b_2+2\cos\frac{2.2s\pi}{n}\,\Sigma^0 b_1b_3+\ldots+2\cos\frac{\frac{1}{2}n.2s\pi}{n}\,\Sigma^0 b_1b_{\frac{1}{2}n+1}\right);$$

and when the order is odd the only difference is that the last terms in the bracketed expressions are

$$2\cos\frac{\frac{1}{2}(n-1)\,2s\pi}{n}\,\Sigma^0 a_1a_{\frac{1}{2}(n+1)} \text{ and } 2\cos\frac{\frac{1}{2}(n-1)\,2s\pi}{n}\,\Sigma^0 b_1b_{\frac{1}{2}(n+1)}$$

respectively.

Beechcroft, Bishopton, N.B.
11*th Sep.* 1881.

TRANSFORMATION OF DIFFERENTIAL EQUATIONS.

By *Sir James Cockle, M.A., F.R.S.*

1. Let

$$D-\beta_1 . D-\beta_2 . u = D-\alpha_1 . D-\alpha_2 . x^2 u \ldots\ldots (1).$$

2. Putting

$$\beta_1-\beta_2, \quad \alpha_1+\alpha_2-\beta_1-\beta_2, \quad \alpha_1-\alpha_2 = I,\ U,\ J,$$

and remembering that transformations to which I have already referred (*ante*, pp. 50–51) enable us to permute I, U, J in any way, and that the order of the α's and β's in I, U, J and (consequently) the signs of I, U, J are indifferent; and that by Boole's algorithm, all, any, or either of the D's in (1) may be augmented or diminished by any (not necessarily the same) even integer, we see that if $I=a$, $U=c$, $J=b$, for instance, represent a soluble case, then, more generally,

$$I,\ U,\ J = 2i \pm a, \quad 2k \pm c, \quad 2j \pm b,$$

where I, U, J may be taken in any order, and i, k, j are any integers positive or negative, will represent a soluble case.

3. This notation is useful for the registration of cases. Thus, recurring to page 50, the forms of (I), (II) and (III) are $I,\ U,\ J = -\frac{2}{3}, -1, -\frac{2}{3}$; $-1, -\frac{8}{3}, \frac{2}{3}$; $1, -\frac{4}{3}, -\frac{2}{3}$; all of which are included in

$$I,\ U,\ J = 2\,(i \pm \tfrac{1}{3}), \quad 2k+1, \quad 2\,(j \pm \tfrac{1}{3}),$$

a form which, together with

$$I,\ U,\ J = 2\,(i \pm \tfrac{1}{3}),\ 2k+1,\ j \pm \tfrac{1}{2},$$

I have elsewhere discussed (*Proc. L. M. S.*, vol. XII.; see p. 67, Art. 50, p. 68, Arts. 51, 53).

4. I add that all Boole's criteria can be expressed in terms of I, U, J; for the four values of $\alpha-\beta$ are comprised in

$$\tfrac{1}{2}\,(U \pm I \pm J).$$

5. When $U=0$ the multiplication of (1) into

$$D-\tfrac{1}{2}\beta_1-\tfrac{1}{2}\beta_2\,(=D-\tfrac{1}{2}\alpha_1-\tfrac{1}{2}\alpha_2)$$

produces a terordinal whereof the symbolical factors on either

side are in arithmetical progression. When $U=4i$ multiplication into $D-\frac{1}{2}\beta_1-\frac{1}{2}\beta_2-2i\,(=D-\frac{1}{2}\alpha_1-\frac{1}{2}\alpha_2)$ will produce a result easily transformed into a terordinal whereof the factors on either side are in arithmetical progression.

6. If in Professor Cayley's equation, cited by Mr. Rawson (*ante*, p. 21), we put $k=x\sqrt{(-1)}$, and change all signs, we get

$$(1+x^2)\,z''+\tfrac{1}{3}\,(1+5x^2)\,\frac{1}{x}z'+\frac{\frac{1}{3}}{1+x^2}z=0.$$

7. Let $u=(1+x^2)^m z$ and we get a binomial in u, provided that $m=\frac{1}{2}$ or $-\frac{1}{6}$.

8. Take $m=\frac{1}{2}$ and we have

$$D\,.\,D-\tfrac{2}{3}\,.\,u+x^2\,.\,D+1\,.\,D+\tfrac{5}{3}\,.\,u=0.$$

9. Take $m=-\frac{1}{6}$ and we have

$$D\,.\,D-\tfrac{2}{3}\,.\,u+x^2\,.\,D+\tfrac{1}{3}\,.\,D-\tfrac{1}{3}\,.\,u=0.$$

10. These equations belong to the form

$$I,\ U,\ J=2\,(i\pm\tfrac{1}{3}),\ 2\,(k\pm\tfrac{2}{3}),\ 2\,(j\pm\tfrac{1}{3}),$$

which differs, in respect of the U, from the first forms of Art. 3. But Mr. Rawson's paper virtually shows that either set is a transformation of the other.

11. I think that it would be as well to start from the quartic

$$y^4+y^2+x_1y-\tfrac{1}{12}=0\ \ldots\ldots\ldots\ldots\ldots\ldots(a),$$

which, though obtained from that of Mr. Rawson by the substitution of 1 for a, is an actual transformation, and not merely a particular case thereof, as will be seen on therein replacing y by $(a)^{\frac{1}{4}}y$ and x_1 by $(a)^{\frac{3}{4}}x_1$.

12. The quartic (a) and that of Professor Cayley are connected by the double transformation

$$y=-i\frac{\sqrt{2}}{\sqrt{3}}\left(\frac{v}{x^{\frac{1}{4}}}+\tfrac{1}{2}\right),\ x_1=i\frac{2\sqrt{2}}{3\sqrt{3}}\left(1-\frac{2}{x^2}\right),$$

where $i=\sqrt{(-1)}$. This comparison made, x_1 in (a) may be replaced by x.

13. I also suggest that Mr. Rawson's (11) should be called the first linear resolvent of his (1). This terminology is expansive as well as expressive. Thus Harley's forms (*Q. J. of Math.*, 1862, vol. v., pp. 351–2) are $(n-1)^{\text{th}}$ linear resolvents, while those of Boole (*Phil. Trans.*, 1864, pp. 735–6) are n^{th} linear resolvents. The class of resolvent is defined by the number of differentiations of the algebraical n-ic equation. The linearity is obtained in the case of a class lower than the $(n-1)^{\text{th}}$ by an exponential substitution.

14. I have given a tolerably complete discussion of what I now call the first linear resolvent of a cubic (see the "Exercises" concluded at p. 119 of vol. VI. and the "Supplement" thereto at pp. 15–18 of vol. X.).

ERRATA.

Old Series, vol. III. p. 249, Art. 88, line 1 *for* "illusion" *read* allusion.
New Series, vol. XI. p. 50, Art. 14, line 1 *for* "XIII" *read* XII.

2, Sandringham Gardens,
Ealing, near London, W.,
October 3, 1881.

ILLUSTRATION OF A THEOREM IN THE THEORY OF EQUATIONS.

By Professor *Cayley*.

THE knowledge of the value of an unsymmetrical function of the roots of a numerical equation adds something to what is given by the equation itself, but it may or may not add anything to what is given by the equation itself in regard to each root separately. If, for instance, α, β, γ being the roots of a cubic equation, it is known that $\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha = a$ given value k, then α, β, γ must denote the roots, taken not in any order whatever, nor yet in a uniquely determinate order, but with a certain restriction as to order, viz. if the roots in a certain order are a, b, c, these roots being such that $a^2b+b^2c+c^2a=k$, then clearly the relation in question $\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha=k$, will be satisfied if α, β, $\gamma = a$, b, c, or $=b$, c, a, or $=c$, a, b (but not if α, β, $\gamma = b$, a, c, or = either of the remaining two arrangements); the relation thus allows α to be $=a$, or $=b$, or $=c$; that is, α is = any one at pleasure of the roots of the cubic equation, and it is thus determined by the cubic equation, and not by any inferior equation; but α being known, the other two roots β and γ will be uniquely, and therefore rationally, determined.

It is worth while to see how the result works out; suppose, for greater simplicity, the cubic equation is $x^3 - 7x + 6 = 0$ roots $(1, 2, -3)$, and that the given relation is $\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha = -1$, then the cubic equation gives

$$\alpha + \beta + \gamma = 0, \quad \alpha\beta + \alpha\gamma + \beta\gamma = -7, \quad \alpha\beta\gamma = -6,$$

and we have, besides, the relation in question

$$\alpha^2\beta + \beta^2\gamma + \gamma^2\alpha = -1;$$

eliminating γ we have

$$\alpha^2 + \alpha\beta + \beta^2 = 7, \quad \alpha\beta(\alpha + \beta) = 6, \quad \alpha^3 + 3\alpha^2\beta - \beta^3 + 1 = 0;$$

or, as it is convenient to write these equations,

$$\beta^2 + \alpha\beta + \alpha^2 - 7 = 0,$$

$$\beta^2 + \alpha\beta - \frac{6}{\alpha} = 0,$$

$$\beta^3 - 3\alpha^2\beta - \alpha^3 - 1 = 0;$$

if from these equations we eliminate β, we obtain two equations in α, which it might be supposed would determine α uniquely; but, by what precedes, α is any root at pleasure of the cubic equation and can thus be determined only by the cubic equation itself, and it follows that any equation obtained by the elimination of β must contain as a factor the cubic function $\alpha^3 - 7\alpha + 6$, and be thus of the form $M(\alpha^3 - 7\alpha + 6) = 0$, where M is a function of α; one result of the elimination is $\alpha^3 - 7\alpha + 6 = 0$, and every other result is of the form just referred to, $M(\alpha^3 - 7\alpha + 6) = 0$; hence we have definitely $\alpha^3 - 7\alpha + 6 = 0$, viz. the roots of the equation $M = 0$ do not apply to the question.

In verification, observe that the first and second equations give $\alpha^2 - 7 = \frac{6}{\alpha}$, that is, $\alpha^2 - 6\alpha + 7 = 0$; to eliminate β from the first and third equations we first find

$$\alpha\beta^2 + (4\alpha^2 - 7)\beta + \alpha^3 + 1 = 0,$$

or say

$$\beta^2 + \left(4\alpha - \frac{7}{\alpha}\right)\beta + \alpha^2 + \frac{1}{\alpha} = 0,$$

and combining herewith the first equation

$$\beta^2 + \alpha\beta + \alpha^2 - 7 = 0,$$

we obtain

$$\beta\left(3\alpha - \frac{7}{\alpha}\right) + 7 + \frac{1}{\alpha} = 0,$$

that is $$\beta = \frac{7\alpha + 1}{-3\alpha^2 + 7},$$

and, substituting in the first equation,

$$(7\alpha + 1)^3 + \alpha(7\alpha + 1)(-3\alpha^2 + 7) + (\alpha^2 - 7)(-3\alpha^2 + 7)^2 = 0,$$

that is

$$\begin{array}{rrrrrrr} & & & 49 & 14 & 1 \\ & -21 & -3 & +49 & +7 & \\ 9 & 0 & -105 & +343 & & -343 \\ \hline 9 & 0 & -126 & -3 & +441 & +21 & -342, \end{array}$$

or dividing by 3

$$3\alpha^6 - 42\alpha^4 - \alpha^3 + 147\alpha^2 + 7\alpha - 114 = 0,$$

which in fact is

$$(\alpha^3 - 7\alpha + 6)(3\alpha^3 - 21\alpha - 19) = 0,$$

of the form in question $M(\alpha^3 - 7\alpha + 6) = 0$. Thus α has any one at pleasure of the three values 1, 2, -3, but α being known we have $\beta = \frac{7\alpha + 1}{-3\alpha^2 + 7}$, and thence $\gamma = -\alpha + \frac{-7\alpha - 1}{-3\alpha^2 + 7}$,

$= \frac{3\alpha^3 - 14\alpha - 1}{-3\alpha^2 + 7}$; in particular $\alpha = 1$, then $\beta = 2$ and $\gamma = -3$.

TRANSACTIONS OF SOCIETIES.

London Mathematical Society.

Thursday, November 10th, S. Roberts, Esq. F.R.S., *President*, in the Chair. The Treasurer's Report showed that the Society was in a very prosperous condition and it was stated that £150 of the current balance had been directed by the Council to be invested in new 3 per cent stock.

The following gentlemen were elected to act as Council for the present Session: *President:* S. Roberts, F.R.S. *Vice-Presidents:* Dr. Hirst, F.R.S.; J. W. L. Glaisher, F.R.S. *Treasurer:* C. W Merrifield, F.R.S. *Hon. Secretaries:* M. Jenkins, M.A.; R. Tucker, M.A. *Other Members:* Prof. Cayley, F.R.S.; Sir J. Cockle, F.R.S.; H. Hart, M.A.; Prof. Henrici, F.R.S.; A. B. Kempe, F.R.S.; Prof. Rowe, M.A.; R. F. Scott, M.A.; Prof. H. J. S. Smith, F.R.S.; H. W. Lloyd Tanner, M.A. and J. J. Walker, M.A.

G. Pirie, M.A., Professor of Mathematics in the University of Aberdeen, was elected a Member. Mr. Merrifield communicated a paper, in the preparation of which he had received assistance from Mr. Jenkins, entitled "Note on the limit to the number of different proper fractions whose denominators are less than x when x is large." If n is the number of such fractions, $n : x^2$ approaches a fixed limit as x increases, thus: $x = 10$, $n : x^2 = \cdot 31$; $x = 100$, $n : x^2 = \cdot 3043$; $x = 1000$, $n : x^2 = \cdot 304191$. The authors arrive at the result $x = \infty$, $n : x^2 = 3 : \pi^2 = \cdot 30396355$. Prof. Cayley F.R.S. read a "Note on Landen's theorem;" Mr. Tucker read portions of papers "On the oscillations of a viscous spheroid," Prof. H. Lamb, M.A., Adelaide; "A geometrical representation of a system of two binary cubics and their associated forms," Mr. W. R. W. Roberts, M.A.; "On the infinitesimal bending of surfaces of revolution," Lord Rayleigh, F.R.S.; "On tangents to a cubic forming a pencil in involution," Mr. R. A. Roberts, B.A.

R. TUCKER, M.A., *Hon. Sec.*

ON TETRAHEDRA WHOSE OPPOSITE EDGES ARE AT RIGHT ANGLES.

By *Ernest Temperley, M.A.*, Fellow of Queens' College, Cambridge.

SUCH a tetrahedron will in this paper be termed an orthogonal tetrahedron. The propositions here proved have been nearly all suggested by analogous properties of triangles, and since an orthogonal tetrahedron is the only one which possesses an orthocentre, we may expect that properties of a plane triangle connected with the orthocentre will have more obvious analogues in the orthogonal tetrahedron than in any other. Moreover, the property of the four perpendiculars of any tetrahedron which corresponds to the concurrence of the three perpendiculars of a plane triangle, viz. that they are generators of the same system of a hyperboloid of one sheet, is not suited to the elementary geometry by which the results of this paper are obtained.

The following notation is used: the corners are denoted by A, B, C, D; α, β, γ are the middle points of the edges BC, CA, AB respectively; and α', β', γ' are the middle points of the opposite edges; the shortest distance between the edges BC, AD meets them in H, H' respectively; whilst K, K'; L, L' are similar points for the pair of edges CA, BD; AB, CD respectively

G_1, G_2, G_3, G_4 are the centres of gravity of the four faces opposite A, B, C, D respectively....,

P_1, P_2, P_3, P_4.....the orthocentres.....,

O_1, O_2, O_3, O_4.....centres of the circumscribing circles of.....,

G the centre of gravity of the tetrahedron.....,

O..... of the circumscribing sphere.

PROP. I. The lines joining the middle points of opposite edges of *any* tetrahedron bisect each other in the centre of gravity, and if the tetrahedron be orthogonal they will be equal to each other.

This is well known, but is introduced here for the sake of completeness.

Since $\beta\gamma$ and $\beta'\gamma'$ are both parallel to BC and equal to half of it, $\beta\gamma\beta'\gamma'$ is a parallelogram, and therefore $\beta\beta'$, $\gamma\gamma'$ bisect each other, and, similarly, $\alpha\alpha'$ is bisected; and in the triangle $A\beta'C$, G_1 divides $\beta'C$ so that $CG_1 = 2G_1\beta'$, and therefore it is easily seen that AG_1 bisects $\beta\beta'$ in a point G, such that $AG = \frac{3}{4}AG_1$, so that this is the centre of gravity

of the tetrahedron. If the tetrahedron be orthogonal, the parallelogram $\beta\gamma\beta'\gamma'$ is a rectangle, and therefore the diagonals are equal. Hence the six middle points of the edges lie on a sphere, whose centre is G; this will be called the *sphere G*.

Prop. II. In an orthogonal tetrahedron the perpendiculars from the vertices on the opposite faces meet in a point, which is collinear with the centre of gravity and the centre of the circumscribing sphere.

Since BC and AD are perpendicular, a plane can be drawn through AD perpendicular to BC, and similarly a plane can be drawn through BC perpendicular to AD; the intersection HH' of these planes is obviously the shortest distance between BC and AD, also AH and DH are both perpendicular to BC, and therefore AH goes through P_4 (fig. 21) the orthocentre of ABC, and the plane ADH contains the perpendicular from D on ABC; similarly a plane BDK contains the same perpendicular, hence DP_4 is the perpendicular.

Also OO_4 is perpendicular to the face ABC, and therefore parallel to DP_4, hence ODP_4O_4 are in one plane, but by plane geometry we know that O_4, G_4, P_4 are collinear, and that $O_4G_4=\frac{1}{2}G_4P_4$; hence DP_4, DG_4, and OO_4 are in one plane. Let OG produced cut DP_4 in P' (fig. 22), draw GJ perpendicular to O_4P_4, then

$$\begin{aligned} P_4J &= \tfrac{3}{4}P_4G_4 \\ &= \tfrac{3}{4}\cdot\tfrac{2}{3}P_4O_4 \\ &= \tfrac{1}{2}P_4O_4, \end{aligned}$$

and therefore J is the middle point of O_4P_4, hence G is the middle point of OP', and therefore OG produced meets DP_4 in a point P', such that $OP'=2OG$; similarly it meets each of the other perpendiculars in the same point. Let this be denoted by P, it may be called the orthocentre.

The first part of this proposition is well known, but I have not seen the collinearity of O, G, P enunciated before.

Prop. III. The six points where the shortest distances between opposite edges meet those edges lie on the sphere through the middle points of the edges.

The sphere through the middle points of the edges cuts each face in the nine-points circle of that face, and therefore passes through the feet of perpendiculars from the vertices of that face upon the opposite sides, but these, by what has been proved, are manifestly the six points H, H', K, K', L, L'.

This was set by me as a problem in the Mathematical

Tripos last January, and this proof of it given by Mr. Lewis in the July number of the *Messenger*. Mr. Lewis proposes for this sphere the name of the Twelve-points Sphere; but, I think, that the next proposition will show that that name should be reserved for another sphere, whose properties present more striking analogies to those of the nine-points circle in plane geometry.

PROP. IV. The sphere through the centres of gravity of the four faces passes through the orthocentres of those faces, and through the points of trisection nearest P of the four lines joining P to the vertices; the radius is one-third of that of the circumscribing sphere; and the centre of gravity and the orthocentre are the internal and external centres of similitude.

Call the circumscribing sphere the sphere O, and that through the centres of gravity of the faces whose centre will be denoted by N call the sphere N.

G_1G_2 is parallel to $\beta'\alpha'$, and

$$= \tfrac{2}{3}\alpha'\beta' = \tfrac{2}{3}.\tfrac{1}{2}AB = \tfrac{1}{3}AB,$$

and similarly for G_2G_3, G_3G_4, &c., hence the tetrahedron $G_1G_2G_3G_4$ is similar and similarly situated to $ABCD$ and of one-third the linear dimensions, hence its circumscribing sphere has its radius equal to $\frac{1}{3}R$ where R is the radius of the sphere O.

Since $GG_1 : GA = GG_2 : GB = \&c. = 1 : 3,$

and G_1, G_2 are on the sphere N, and A, B on the sphere O; therefore G is the internal centre of similitude of the two spheres; hence the centre N lies on OG produced so that

$$OG : GN = 3 : 1,$$

and therefore $$ON = \tfrac{4}{3}OG = \tfrac{2}{3}OP,$$

therefore $$PN = \tfrac{1}{3}PO,$$

and P is the external centre of similitude.

Let a, b, c, d be the points of trisection nearest P of the lines PA, PB, PC, PD respectively, then by the property of centres of similitude the sphere N goes through a, b, c, d. To show that it also goes through $P_1P_2P_3P_4$.

In fig. 22 join G_4N and produce it to cut PD in d', and draw G_4Q parallel to PD to meet PO in Q; then

$$GQ : GO = JG_4 : JO_4 = \tfrac{1}{6}P_4O_4 : \tfrac{1}{2}P_4O_4 = 1 : 3;$$

therefore

$$GQ = \tfrac{1}{6}PO \text{ and } NG = PG - PN = \tfrac{1}{2}PO - \tfrac{1}{3}PO = \tfrac{1}{6}PO;$$

therefore $NQ = \frac{1}{3}PO = PN$;

therefore $Pd' = G_4Q = \frac{1}{3}PD = Pd$;

therefore d' coincides with d. Hence G_4d is a diameter of the sphere N, which consequently passes through P_4; similarly it goes through $P_1P_2P_3$.

Cor. This sphere passes through the circles described on G_1P_1, G_2P_2, &c., as diameters and lying on the faces of the tetrahedron.

PROP. V. The orthocentre of an orthogonal tetrahedron is the orthocentre of the four triangles, in which the tetrahedron is cut by the planes through the edges perpendicular to the opposite edges.

It is obvious from the proof of Prop. II. that the triangle AHD has P for its orthocentre, whence the result.

Cor. The rectangle

$$PD.PP_4 = PA.PP_1 = PH.PH' = \&c.$$

PROP. VI. An orthogonal tetrahedron has a sphere with respect to which it is self-conjugate.

For since

$$PA.PP_1 = PB.PP_2 = PC.PP_3 = PD.PP_4,$$

it is clear that if with centre P and radius ρ, such that $\rho^2 = PA.PP_1$ a sphere be described, then A and the plane BCD are pole and polar, and similarly for the rest.

In fig. 21 where P is inside the tetrahedron, the rectangle $PA.PP_1$ is negative, and therefore ρ imaginary. This sphere may be called the sphere P.

PROP. VII. The radius ρ of the sphere P is given by the equation

$$3\rho^2 = OP^2 - R^2.$$

For $\rho^2 = PA.PP_1 = PD\ PP_4$ (fig. 22),

and $PD = 3Pd$;

therefore $\rho^2 = 3Pd.PP_4$,

and d and P_4 are on the sphere N, whose centre is N and radius $\frac{1}{3}R$; therefore

$$Pd.PP_4 = PN^2 - (\tfrac{1}{3}R)^2 = (\tfrac{1}{3}OP)^2 - \tfrac{1}{9}R^2;$$

whence $\rho^2 = \frac{1}{3}(OP^2 - R^2)$.

Cor. The sphere P is real or imaginary according as OP

is greater or less than R, *i.e.* according as the orthocentre is outside or inside the circumscribing sphere.

PROP. VIII. The radius ρ' of the sphere G is given by

$$3\rho'^2 = R^2 - \tfrac{1}{4}OP^2 = R^2 - PG^2.$$

For $\rho'^2 - GP^2 =$ rectangle of segments of chords of sphere G, which pass through P

$$= - PH.PH' = - PD.PP_4 \text{ (by PROP. V)};$$

therefore $$\rho'^2 - GP^2 = -\rho^2 \text{ (1) (PROP. VI)}$$

$$= \tfrac{1}{3}(R^2 - OP^2),$$

whence $$3\rho'^2 = R^2 - \tfrac{1}{4}OP^2 = R^2 - PG^2.$$

PROP. IX. The spheres G and P cut orthogonally.

This follows at once from equation (1) in PROP. VIII; for

$$\rho^2 + \rho'^2 = PG^2.$$

PROP. X. The four spheres O, G, N, P have a common radical plane, which cuts OP in a point T, such that

$$3\,OP.OT = OP^2 + 2R^2.$$

The radical plane of the spheres O and P cuts OP in a point T_1, such that

$$OT_1^2 - PT_1^2 = R^2 - \rho^2,$$

or $$(OT_1 - PT_1)(OT_1 + PT_1) = R^2 - \tfrac{1}{3}(OP^2 - R^2),$$

$$OP(OT_1 + OT_1 - OP) = \tfrac{4}{3}R^2 - \tfrac{1}{3}OP^2,$$

whence $$3\,OP.OT_1 = 2R^2 + OP^2;$$

and therefore T_1 coincides with T.

Next let the radical planes of G and P, and of N and P cut OP in T_2 and T_3, we have

$$GT_2^2 - PT_2^2 = \rho'^2 - \rho^2 = \tfrac{1}{3}R^2 - \tfrac{1}{12}OP^2 - \tfrac{1}{3}OP^2 + \tfrac{1}{3}R^2,$$

or $$(GT_2 - PT_2)(GT_2 + PT_2) = \tfrac{1}{3}R^2 - \tfrac{5}{12}OP^2,$$

and $$GT_2 = OT_2 - OG = OT_2 - \tfrac{1}{2}OP,$$

$$PT_2 = OT_2 - OP;$$

whence we get $$3\,OT_2.OP = 2R^2 + OP^2;$$

and therefore T_2 coincides with T.

Also $$NT_3^2 - PT_3^2 = \tfrac{1}{9}R^2 - \rho^2;$$

whence also T_3 coincides with T.

From these propositions many results may be deduced; a few may be noticed here.

In fig. 22 $QG_4 = Pd = \frac{1}{3}PD$,

hence

$$QG_4 . PP_4 = \tfrac{1}{3}PD . PP_4 = -\tfrac{1}{3}\rho^2 = QG_1 . PP_1 = \&c.;$$

and therefore if with P and Q as foci we describe a quadric of revolution, the square of whose semi-minor axis is $-\frac{1}{3}\rho^2$, this quadric will touch the four faces of the tetrahedron. Its auxiliary sphere is evidently the sphere N; and therefore the major axis of the quadric is $\frac{2}{3}R$; it will be an ellipsoid when ρ is imaginary.

Again, if $ABCD$ be an orthogonal tetrahedron, so also is $PABC$, and D is then its orthocentre; hence our propositions may be applied to the tetrahedron formed by the orthocentre of the given one and any three vertices, the fourth vertex being in that case the orthocentre.

Hence, applying PROP. III to the tetrahedron $PABC$, we see that the sphere through the middle points of its edges passes also through $P_1P_2P_3$, and we get the following theorem: A sphere will pass through the nine-points circle of any face and through the orthocentres of the three remaining faces, and this sphere bisects the lines joining the orthocentre of the tetrahedron to the three vertices of the first face.

It may be readily proved that the radii of these four spheres are all equal, and equal also to that of the sphere through the middle points of the edges of the given tetrahedron.

Next, take PROP. IV, and consider the tetrahedron $PABC$, the orthocentres of the three faces PBC, PCA, PAB are evidently H', K', L'; and therefore the sphere through H', K', L', P_4 goes through the points of trisection nearest D of the sides DA, DB, DC, &c., and we can enunciate the theorem:

If in an orthogonal tetrahedron a sphere be described through the orthocentre of any face and the feet of the perpendiculars from the orthocentre of the tetrahedron on the three edges which do not lie on the given face, it will pass also through the three points of trisection of those edges which are nearest their intersection, and through the centres of gravity of the four faces of the tetrahedron, whose vertex is the orthocentre of the given tetrahedron and base the given triangle.

It may be mentioned that neither of the spheres G or N touch the inscribed sphere.

ON ELLIPTIC FUNCTIONS.

II.

(Continued from p. 95).

By *J. W. L. Glaisher.*

Transformations, §§ 16, 17.

§ 16. THE changes produced in the twelve elliptic functions when the argument and modulus are replaced by iu and k', by ku and $\frac{1}{k}$, and by $k'u$ and $\frac{ik}{k'}$ respectively are shown in the following table, in which the notation of § 5 (p. 85) is used.

	$v = iu,\ a = k'.$	$v = ku,\ a = \frac{1}{k}.$	$v = k'u,\ a = \frac{ik}{k'}.$
sn (v, a)	i sc u	k sn u	k' sd u
cn (v, a)	nc u	dn u	cd u
dn (v, a)	dc u	cn u	nd u
ns (v, a)	$-i$ cs u	$\frac{1}{k}$ ns u	$\frac{1}{k'}$ ds u
nc (v, a)	cn u	nd u	dc u
nd (v, a)	cd u	nc u	dn u
sc (v, a)	i sn u	k sd u	k' sc u
sd (v, a)	i sd u	k sc u	k' sn u
cd (v, a)	nd u	dc u	cn u
cs (v, a)	$-i$ ns u	$\frac{1}{k}$ ds u	$\frac{1}{k'}$ cs u
ds (v, a)	$-i$ ds u	$\frac{1}{k}$ cs u	$\frac{1}{k'}$ ns u
dc (v, a)	dn u	cd u	nc u

The table shows, for example, that

$$\text{sn}\,(iu, k') = i\,\text{sc}\,u,\quad \text{sn}\left(ku, \frac{1}{k}\right) = k\,\text{sn}\,u,\quad \text{sn}\left(k'u, \frac{ik}{k'}\right) = k'\,\text{sd}\,u,$$

and so on.

§ 17. The table in the preceding section may be expressed also in the form

	$v = iu,\ a = k'.$	$v = ku,\ a = \frac{1}{k}.$	$v = k'u,\ a = \frac{ik}{k'}.$
$\mathrm{sn}\,(v, a)$	$\frac{1}{k'}\,\mathrm{dn}\,(u+K+iK')$	$k\,\mathrm{sn}\,u$	$-\,\mathrm{cn}\,(u+K)$
$\mathrm{cn}\,(v, a)$	$\frac{ik}{k'}\,\mathrm{cn}\,(u+K+iK')$	$\mathrm{dn}\,u$	$\mathrm{sn}\,(u+K)$
$\mathrm{dn}\,(v, a)$	$k\ \mathrm{sn}\,(u+K+iK')$	$\mathrm{cn}\,u$	$\frac{1}{k'}\,\mathrm{dn}\,(u+K)$
$\mathrm{ns}\,(v, a)$	$-i\,\mathrm{cs}\,u$	$\frac{1}{k}\,\mathrm{ns}\,u$	$\frac{1}{k'}\,\mathrm{ds}\,u$
$\mathrm{ds}\,(v, a)$	$-i\,\mathrm{ds}\,u$	$\frac{1}{k}\,\mathrm{cs}\,u$	$\frac{1}{k'}\,\mathrm{ns}\,u$
$\mathrm{cs}\,(v, a)$	$-i\,\mathrm{ns}\,u$	$\frac{1}{k}\,\mathrm{ds}\,u$	$\frac{1}{k'}\,\mathrm{cs}\,u$
$\mathrm{dc}\,(v, a)$	$-ik'\,\mathrm{sc}\,(u+K+iK')$	$\frac{1}{k}\,\mathrm{dc}\,(u+iK')$	$\mathrm{nc}\,u$
$\mathrm{nc}\,(v, a)$	$-\frac{ik'}{k}\,\mathrm{nc}\,(u+K+iK')$	$-i\,\mathrm{sc}\,(u+iK')$	$\mathrm{dc}\,u$
$\mathrm{sc}\,(v, a)$	$-\frac{i}{k}\,\mathrm{dc}\,(u+K+iK')$	$-i\,\mathrm{nc}\,(u+iK')$	$k'\,\mathrm{sc}\,u$
$\mathrm{cd}\,(v, a)$	$\mathrm{nd}\,u$	$k\,\mathrm{cd}\,(u+iK')$	$k'\,\mathrm{sd}\,(u+K)$
$\mathrm{sd}\,(v, a)$	$i\,\mathrm{sd}\,u$	$-ik\,\mathrm{nd}\,(u+iK')$	$-k'\,\mathrm{cd}\,(u+K)$
$\mathrm{nd}\,(v, a)$	$\mathrm{cd}\,u$	$-ik\,\mathrm{sd}\,(u+iK')$	$k'\,\mathrm{nd}\,(u+K)$

In this arrangement the twelve functions are regarded as forming four distinct groups, viz.

(i) $\mathrm{sn}\,u,\ \mathrm{cn}\,u,\ \mathrm{dn}\,u,$

(ii) $\mathrm{ns}\,u,\ \mathrm{ds}\,u,\ \mathrm{cs}\,u,$

(iii) $\mathrm{dc}\,u,\ \mathrm{nc}\,u,\ \mathrm{sc}\,u,$

(iv) $\mathrm{cd}\,u,\ \mathrm{sd}\,u,\ \mathrm{nd}\,u,$

each of which is convertible with the other three by the substitution of $u+K$, of $u+iK'$ and of $u+K+iK'$ for u (§ 10, p. 90).

This division of the functions into groups having the same denominator is generally preferable to the arbitrary arrangement followed in §§ 3, 8, 9, and in the last section, in which they are written in the order $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$, and their reciprocals, $\mathrm{sc}\,u$, $\mathrm{sd}\,u$, $\mathrm{cd}\,u$, and their reciprocals.

System of differential formulæ, §§ 18–20.

§ 18. It is easy to prove by differentiation that we have the following system of formulæ:

$$\frac{d^2}{du^2}\mathrm{sn}^n u = n(n-1)\ \mathrm{sn}^{n-2}u - n^2(1+k^2)\,\mathrm{sn}^n u + n(n+1)k^2\,\mathrm{sn}^{n+2}u,$$

$$\frac{d^2}{du^2}\mathrm{cn}^n u = n(n-1)k'^2\,\mathrm{cn}^{n-2}u + n^2(k^2-k'^2)\,\mathrm{cn}^n u - n(n+1)k^2\,\mathrm{cn}^{n+2}u,$$

$$\frac{d^2}{du^2}\mathrm{dn}^n u = -n(n-1)k'^2\,\mathrm{dn}^{n-2}u + n^2(1+k'^2)\,\mathrm{dn}^n u - n(n+1)\ \mathrm{dn}^{n+2}u,$$

$$\frac{d^2}{du^2}\mathrm{ns}^n u = n(n-1)k^2\ \mathrm{ns}^{n-2}u - n^2(1+k^2)\,\mathrm{ns}^n u + n(n+1)\,\mathrm{ns}^{n+2}u,$$

$$\frac{d^2}{du^2}\mathrm{ds}^n u = -n(n-1)k^2k'^2\,\mathrm{ds}^{n-2}u + n^2(k^2-k'^2)\,\mathrm{ds}^n u + n(n+1)\,\mathrm{ds}^{n+2}u,$$

$$\frac{d^2}{du^2}\mathrm{cs}^n u = n(n-1)k'^2\ \mathrm{cs}^{n-2}u + n^2(1+k'^2)\,\mathrm{cs}^n u + n(n+1)\,\mathrm{cs}^{n+2}u,$$

$$\frac{d^2}{du^2}\mathrm{dc}^n u = n(n-1)k^2\,\mathrm{dc}^{n-2}u - n^2(1+k^2)\,\mathrm{dc}^n u + n(n+1)\ \mathrm{dc}^{n+2}u,$$

$$\frac{d^2}{du^2}\mathrm{nc}^n u = -n(n-1)k^2\,\mathrm{nc}^{n-2}u + n^2(k^2-k'^2)\,\mathrm{nc}^n u + n(n+1)k'^2\,\mathrm{nc}^{n+2}u,$$

$$\frac{d^2}{du^2}\mathrm{sc}^n u = n(n-1)\ \mathrm{sc}^{n-2}u + n^2(1+k'^2)\,\mathrm{sc}^n u + n(n+1)k'^2\,\mathrm{sc}^{n+2}u,$$

$$\frac{d^2}{du^2}\mathrm{cd}^n u = n(n-1)\,\mathrm{cd}^{n-2}u - n^2(1+k^2)\,\mathrm{cd}^n u + n(n+1)k^2\ \mathrm{cd}^{n-2}u,$$

$$\frac{d^2}{du^2}\mathrm{sd}^n u = n(n-1)\,\mathrm{sd}^{n-2}u + n^2(k^2-k'^2)\,\mathrm{sd}^n u - n(n+1)k^2k'^2\,\mathrm{sd}^{n+2}u,$$

$$\frac{d^2}{du^2}\mathrm{nd}^n u = -n(n-1)\,\mathrm{nd}^{n-2}u + n^2(1+k'^2)\,\mathrm{nd}^n u - n(n+1)k'^2\ \mathrm{nd}^{n+2}u.$$

§ 19. Since $\operatorname{ns} u$, $\operatorname{cs} u$, &c. are the reciprocals of $\operatorname{sn} u$, $\operatorname{sc} u$, &c. six of these formulæ are deducible from the other six by changing the sign of n. It will be observed that the coefficients $n(n-1)$, n^2, $n(n+1)$ are such that their order is reversed by a change of sign of n.

The formulæ in any three of the groups may be deduced from those in the fourth group by the substitution of $u+K$, of $u+iK$, and of $u+K+iK'$ for u (§ 10, p. 90).

Any two formulæ of any one of the four groups may be deduced from the third formula by means of the transformations in the last section; for, considering for example the first formula in the first group, we have, by replacing u and k by iu and k', and by $k'u$ and $\frac{ik}{k'}$, the equations

$$\frac{1}{i^2}\frac{d^2}{du^2}\operatorname{sn}^n(iu,\ k') = n(n-1)\operatorname{sn}^{n-2}(iu,\ k') - n^2(1+k'^2)\operatorname{sn}^n(iu,\ k') + n(n+1)\,k'^2\operatorname{sn}^{n+2}(iu,\ k'),$$

$$\frac{1}{k'^2}\frac{d^2}{du^2}\operatorname{sn}^n\left(k'u,\ \frac{ik}{k'}\right) = n(n-1)\operatorname{sn}^{n-2}\left(k'u,\ \frac{ik}{k'}\right) - n^2\left(1-\frac{k^2}{k'^2}\right)\operatorname{sn}^n\left(k'u,\ \frac{ik}{k'}\right) - n(n+1)\frac{k^2}{k'^2}\operatorname{sn}^{n+2}\left(k'u,\ \frac{ik}{k'}\right),$$

which by means of the first and third results in the first line of the table in § 17, viz.

$$\operatorname{sn}(iu,\ k') = \frac{1}{k'}\operatorname{dn}(u+K+iK'), \quad \operatorname{sn}\left(k'u,\ \frac{ik}{k'}\right) = -\operatorname{cn}(u+K),$$

give the third and second formulæ of the group on replacing $u+K+iK'$ and $u+K$ by u.

It thus appears that from any one of the twelve formulæ the other eleven may be deduced by means of the transformations in the last section, and the substitution of $u+K$, of $u+iK'$, or of $u+K+iK'$ for u.

It may be remarked that the transformations in §17 are such that by the first sn and dn are interchanged, cn remaining unchanged; by the second cn and dn are interchanged, sn remaining unchanged; and by the third sn and cn are interchanged, dn remaining unchanged; and a similar remark applies to the other three groups.

Of the four groups the second, which involves the functions ns, ds, cs is the most regular.

§ 20. The formulæ of § 18, arranged in a different order as in § 17, so that the members of each group are convertible into one another by changes of the argument only, may be written as follows. In each of the formulæ the quantity in { } is, for the sake of brevity, expressed only once.

$$\frac{d^2}{du^2}\{k\ \operatorname{sn}u\}^n = n(n-1)k^2\ \{\ \}^{n-2} - n^2(1+k^2)\{\ \}^n + n(n+1)\{\ \}^{n+2},$$

$$\frac{d^2}{du^2}\{\ \operatorname{ns}u\}^n = n(n-1)k^2\ \{\ \}^{n-2} - n^2(1+k^2)\{\ \}^n + n(n+1)\{\ \}^{n+2},$$

$$\frac{d^2}{du^2}\{\ \operatorname{dc}u\}^n = n(n-1)k^2\ \{\ \}^{n-2} - n^2(1+k^2)\{\ \}^n + n(n+1)\{\ \}^{n+2},$$

$$\frac{d^2}{du^2}\{k\ \operatorname{cd}u\}^n = n(n-1)k^2\ \{\ \}^{n-2} - n^2(1+k^2)\{\ \}^n + n(n+1)\{\ \}^{n+2},$$

$$\frac{d^2}{du^2}\{k\ \operatorname{cn}u\}^n = n(n-1)k^2k'^2\{\ \}^{n-2} + n^2(k^2-k'^2)\{\ \}^n - n(n+1)\{\ \}^{n+2},$$

$$\frac{d^2}{du^2}\{\ \operatorname{ds}u\}^n = -n(n-1)k^2k'^2\{\ \}^{n-2} + n^2(k^2-k'^2)\{\ \}^n + n(n+1)\{\ \}^{n-2},$$

$$\frac{d^2}{du^2}\{k'\ \operatorname{nc}u\}^n = -n(n-1)k^2k'^2\{\ \}^{n-2} + n^2(k^2-k'^2)\{\ \}^n + n(n+1)\{\ \}^{n+2},$$

$$\frac{d^2}{du^2}\{kk'\operatorname{sd}u\}^n = n(n-1)k^2k'^2\{\ \}^{n-2} + n^2(k^2-k'^2)\{\ \}^n - n(n+1)\{\ \}^{n+2},$$

$$\frac{d^2}{du^2}\{\ \operatorname{dn}u\}^n = -n(n-1)\ k'^2\{\ \}^{n-2} + n^2(1+k'^2)\{\ \}^n - n(n+1)\{\ \}^{n+2},$$

$$\frac{d^2}{du^2}\{\ \operatorname{cs}u\}^n = n(n-1)\ k'^2\{\ \}^{n-2} + n^2(1+k'^2)\{\ \}^n + n(n+1)\{\ \}^{n+2},$$

$$\frac{d^2}{du^2}\{k'\ \operatorname{sc}u\}^n = n(n-1)\ k'^2\{\ \}^{n-2} + n^2(1+k'^2)\{\ \}^n + n(n+1)\{\ \}^{n+2},$$

$$\frac{d^2}{du^2}\{k'\ \operatorname{nd}u\}^n = -n(n-1)\ k'^2\{\ \}^{n-2} + n^2(1+k'^2)\{\ \}^n - n(n+1)\{\ \}^{n+2}.$$

§ 21. From the differential formulæ in § 18 we obtain by integration the following system of formulæ of reduction for $\int \mathrm{sn}^n u\,du$, &c.

$$
\begin{aligned}
&(n+1)k^2 \quad \int \mathrm{sn}^{n+2}u\,du - n(1+k^2)\int \mathrm{sn}^n u\,du + (n-1)\int \mathrm{sn}^{n-2}u\,du - \mathrm{sn}^{n-1}u\,\mathrm{cn}u\,\mathrm{dn}u = 0,\\
&(n+1)k^2 \quad \int \mathrm{cn}^{n+2}u\,du - n(k^2-k'^2)\int \mathrm{cn}^n u\,du - (n-1)k'^2\int \mathrm{cn}^{n-2}u\,du - \mathrm{cn}^{n-1}u\,\mathrm{dn}u\,\mathrm{sn}u = 0,\\
&(n+1) \quad \int \mathrm{dn}^{n+2}u\,du - n(1+k'^2)\int \mathrm{dn}^n u\,du + (n-1)k'^2\int \mathrm{dn}^{n-2}u\,du - k^2\,\mathrm{dn}^{n-1}u\,\mathrm{sn}u\,\mathrm{cn}u = 0,\\
&(n+1) \quad \int \mathrm{ns}^{n+2}u\,du - n(1+k^2)\int \mathrm{ns}^n u\,du + (n-1)k^2\int \mathrm{ns}^{n-2}u\,du + \mathrm{ns}^{n-1}u\,\mathrm{ds}u\,\mathrm{cs}u = 0,\\
&(n+1) \quad \int \mathrm{ds}^{n+2}u\,du + n(k^2-k'^2)\int \mathrm{ds}^n u\,du - (n-1)k^2k'^2\int \mathrm{ds}^{n-2}u\,du + \mathrm{ds}^{n-1}u\,\mathrm{cs}u\,\mathrm{ns}u = 0,\\
&(n+1) \quad \int \mathrm{cs}^{n+2}u\,du + n(1+k'^2)\int \mathrm{cs}^n u\,du + (n-1)k'^2\int \mathrm{cs}^{n-2}u\,du + \mathrm{cs}^{n-1}u\,\mathrm{ns}u\,\mathrm{ds}u = 0,\\
&(n+1) \quad \int \mathrm{dc}^{n+2}u\,du - n(1+k^2)\int \mathrm{dc}^n u\,du + (n-1)k^2\int \mathrm{dc}^{n-2}u\,du - k'^2\,\mathrm{dc}^{n-1}u\,\mathrm{nc}u\,\mathrm{sc}u = 0,\\
&(n+1)k'^2 \quad \int \mathrm{nc}^{n+2}u\,du + n(k^2-k'^2)\int \mathrm{nc}^n u\,du - (n-1)k^2\int \mathrm{nc}^{n-2}u\,du - \mathrm{nc}^{n-1}u\,\mathrm{sc}u\,\mathrm{dc}u = 0,\\
&(n+1)k'^2 \quad \int \mathrm{sc}^{n+2}u\,du + n(1+k'^2)\int \mathrm{sc}^n u\,du + (n-1)\int \mathrm{sc}^{n-2}u\,du - \mathrm{sc}^{n-1}u\,\mathrm{dc}u\,\mathrm{nc}u = 0,\\
&(n+1)k^2 \quad \int \mathrm{cd}^{n+2}u\,du - n(1+k^2)\int \mathrm{cd}^n u\,du + (n-1)\int \mathrm{cd}^{n-2}u\,du + k'^2\,\mathrm{cd}^{n-1}u\,\mathrm{sd}u\,\mathrm{nd}u = 0,\\
&(n+1)k^2k'^2 \int \mathrm{sd}^{n+2}u\,du - n(k^2-k'^2)\int \mathrm{sd}^n u\,du - (n-1)\int \mathrm{sd}^{n-2}u\,du + \mathrm{sd}^{n-1}u\,\mathrm{nd}u\,\mathrm{cd}u = 0,\\
&(n+1)k'^2 \quad \int \mathrm{nd}^{n+2}u\,du - n(1+k'^2)\int \mathrm{nd}^n u\,du + (n-1)\int \mathrm{nd}^{n-2}u\,du + k^2\,\mathrm{nd}^{n-1}u\,\mathrm{cd}u\,\mathrm{sd}u = 0.
\end{aligned}
$$

These formulæ are connected with one another in the same manner as the differential formulæ of § 18 from which they have been derived. The relations between them are therefore the same as those stated in § 19.

§22. These formula of reduction may be written also as follows:

$$(n+1)\int (k\,\mathrm{sn}\,u)^{n+2}du-n(1+k^2)\int (k\,\mathrm{sn}\,u)^n du+(n-1)k^2\int (k\,\mathrm{sn}\,u)^{n-2}du-k^n\,\mathrm{sn}^{n-1}u\,\mathrm{cn}\,u\,\mathrm{dn}\,u=0,$$
$$(n+1)\int (\mathrm{ns}\,u)^{n+2}du-n(1+k^2)\int (\mathrm{ns}\,u)^n du+(n-1)k^2\int (\mathrm{ns}\,u)^{n-2}du+\mathrm{ns}^{n-1}u\,\mathrm{ds}\,u\,\mathrm{cs}\,u=0,$$
$$(n+1)\int (\mathrm{dc}\,u)^{n+2}du-n(1+k^2)\int (\mathrm{dc}\,u)^n du+(n-1)k^2\int (\mathrm{dc}\,u)^{n-2}du-k'^2\,\mathrm{dc}^{n-1}u\,\mathrm{nc}\,u\,\mathrm{sc}\,u=0,$$
$$(n+1)\int (k\,\mathrm{cd}\,u)^{n+2}du-n(1+k^2)\int (k\,\mathrm{cd}\,u)^n du+(n-1)k^2\int (k\,\mathrm{cd}\,u)^{n-2}du+k^nk'^2\,\mathrm{cd}^{n-1}u\,\mathrm{sd}\,u\,\mathrm{nd}u=0,$$

$$(n+1)\int (k\,\mathrm{cn}\,u)^{n+2}du-n(k^2-k'^2)\int (k\,\mathrm{cn}\,u)^n du-(n-1)k^2k'^2\int (k\,\mathrm{cn}\,u)^{n-2}du-k^n\,\mathrm{cn}^{n-1}u\,\mathrm{dn}u\,\mathrm{sn}\,u=0,$$
$$(n+1)\int (\mathrm{ds}\,u)^{n+2}du+n(k^2-k'^2)\int (\mathrm{ds}\,u)^n du-(n-1)k^2k'^2\int (\mathrm{ds}\,u)^{n-2}du+\mathrm{ds}^{n-1}u\,\mathrm{cs}\,u\,\mathrm{ns}\,u=0,$$
$$(n+1)\int (k'\,\mathrm{nc}\,u)^{n+2}du+n(k^2-k'^2)\int (k'\,\mathrm{nc}\,u)^n du-(n-1)k^2k'^2\int (k'\,\mathrm{nc}\,u)^{n-2}du-k'^n\,\mathrm{nc}^{n-1}u\,\mathrm{sc}u\,\mathrm{dc}\,u=0,$$
$$(n+1)\int (kk'\,\mathrm{sd}\,u)^{n+2}du-n(k^2-k'^2)\int (kk'\,\mathrm{sd}\,u)^n du-(n-1)k^2k'^2\int (kk'\,\mathrm{sd}\,u)^{n-2}du+k^nk'^n\,\mathrm{sd}^{n-1}u\,\mathrm{nd}\,u\,\mathrm{cd}\,u=0,$$

$$(n+1)\int (\mathrm{dn}u)^{n+2}du-n(1+k'^2)\int (\mathrm{dn}u)^n du+(n-1)k'^2\int (\mathrm{dn}u)^{n-2}du-k^2\,\mathrm{dn}^{n-1}u\,\mathrm{sn}\,u\,\mathrm{cn}\,u=0,$$
$$(n+1)\int (\mathrm{cs}\,u)^{n+2}du+n(1+k'^2)\int (\mathrm{cs}\,u)^n du+(n-1)k'^2\int (\mathrm{cs}\,u)^{n-2}du+\mathrm{cs}^{n-1}u\,\mathrm{ns}u\,\mathrm{ds}\,u=0,$$
$$(n+1)\int (k'\,\mathrm{sc}\,u)^{n+2}du+n(1+k'^2)\int (k'\,\mathrm{sc}\,u)^n du+(n-1)k'^2\int (k'\,\mathrm{sc}\,u)^{n-2}du-k'^n\,\mathrm{sc}^{n-1}u\,\mathrm{dc}u\,\mathrm{nc}\,u=0,$$
$$(n+1)\int (k'\,\mathrm{nd}u)^{n+2}du-n(1+k'^2)\int (k'\,\mathrm{nd}u)^n du+(n-1)k'^2\int (k'\,\mathrm{nd}\,u)^{n-2}du+k^2k'^n\,\mathrm{nd}^{n-1}u\,\mathrm{cd}\,u\,\mathrm{sd}\,u=0.$$

This system of formulæ corresponds to that given in §20, from which it is immediately deducible by integration.

§ 23. If $\phi(u)$ denotes any one of the twelve elliptic functions, then, by means of the formulæ in § 21, $\int\phi^n(u)\,du$ may be reduced to dependence upon $\int\phi^2(u)\,du$ or upon $\int\phi(u)\,du$ according as n is an even or an uneven positive integer. The twelve integrals $\int \mathrm{sn}u\,du$, &c. are expressible in terms of elliptic functions, and the twelve integrals $\int \mathrm{sn}^2u\,du$, are expressible either in terms of Jacobi's function $\mathrm{Z}(u)$ or in terms of this function and of elliptic functions. It follows therefore that $\int\phi^n(u)\,du$ is expressible in terms of the zeta function or of this function and elliptic functions if n is an even integer, and in terms of elliptic functions only, if n is an uneven integer.

Trigonometrical formulæ of reduction, § 24.

§ 24. By putting $k=0$ in the formulæ of § 21, we find

$$n\int\sin^n u\,du-(n-1)\int\sin^{n-2}u\,du+\sin^{n-1}u\cos u=0,$$

$$n\int\cos^n u\,du-(n-1)\int\cos^{n-2}u\,du-\cos^{n-1}u\sin u=0,$$

$$(n+1)\int\mathrm{cosec}^{n+2}u\,du-n\int\mathrm{cosec}^n u\,du+\mathrm{cosec}^{n+1}u\cos u=0,$$

$$(n+1)\int\sec^{n+2}u\,du\quad-n\int\sec^n u\,du\quad-\sec^{n+1}u\quad\sin u=0,$$

$$(n+1)\int\tan^{n+2}u\,du+2n\int\tan^n u\,du$$
$$+(n-1)\int\tan^{n-2}u\,du-\tan^{n-1}u\sec^2u\quad=0,$$

$$(n+1)\int\cot^{n+2}u\,du+2n\int\cot^n u\,du$$
$$+(n-1)\int\cot^{n-2}u\,du+\cot^{n-1}u\,\mathrm{cosec}^2u=0.$$

The first four of these equations are the formulæ of reduction used in Trigonometry; but the last two are replaced by the more simple formulæ

$$(n+1)\int\tan^{n+2}u\,du+(n+1)\int\tan^n u\,du-\tan^{n+1}u=0,$$

$$(n+1)\int\cot^{n+2}u\,du+(n+1)\int\cot^n u\,du+\cot^{n+1}u=0.$$

Values of $\int \mathrm{sn}\, u\, du$, &c. *and of* $\int \mathrm{sn}^2 u\, du$, &c., §§ 25–30.

§ 25. The values of $\int \mathrm{sn}\, u\, du$, &c. have been already given in several forms in §§ 14 and 15. If we select only the real forms and arrange the results in groups as in § 21 we obtain the following system of formulæ.

$$\int \mathrm{sn}u\, du = \frac{1}{k} \log (\mathrm{dn}u - k\, \mathrm{cn}u),$$

$$\int \mathrm{cn}u\, du = \frac{1}{k} \arctan \left(k\, \frac{\mathrm{sn}u}{\mathrm{dn}u} \right),$$

$$\int \mathrm{dn}u\, du = \arctan \left(\frac{\mathrm{sn}u}{\mathrm{cn}u} \right),$$

$$\int \mathrm{ns}u\, du = -\log (\mathrm{ds}u + \mathrm{cs}u),$$

$$\int \mathrm{ds}u\, du = \log (\mathrm{ns}u - \mathrm{cs}u),$$

$$\int \mathrm{cs}u\, du = \log (\mathrm{ns}u - \mathrm{ds}u),$$

$$\int \mathrm{dc}u\, du = -k' \log (\mathrm{nc}u - \mathrm{sc}u),$$

$$\int \mathrm{nc}u\, du = \frac{1}{k'} \log (\mathrm{dc}u + k'\, \mathrm{sc}u),$$

$$\int \mathrm{sc}u\, du = -\frac{1}{k'} \log (\mathrm{dc}u - k'\, \mathrm{nc}u),$$

$$\int \mathrm{cd}u\, du = \frac{1}{k} \log (\mathrm{nd}u + k\, \mathrm{sd}u),$$

$$\int \mathrm{sd}u\, du = \frac{1}{kk'} \arctan \left(\frac{k'}{k}\, \frac{\mathrm{nd}u}{\mathrm{cd}u} \right),$$

$$\int \mathrm{nd}u\, du = \frac{1}{k'} \arctan \left(k'\, \frac{\mathrm{sd}u}{\mathrm{cd}u} \right).$$

§ 26. The corresponding values of $\int \mathrm{sn}^2 u\,du$, &c. are given by the following system of formulæ. The results are expressed in terms of the function $Z(u)$, which is defined by the equation

$$Z(u) = E(u) - \frac{E}{K}u,$$

where $E(u) = \int_0^u \mathrm{dn}^2 u\,du = \int_0^\phi \Delta\phi\,d\phi$ if $\phi = \mathrm{am}\,u$.

$$\int \mathrm{sn}^2 u\,du = \frac{K-E}{k^2K}u - \frac{\mathrm{Z}(u)}{k^2},$$

$$\int \mathrm{cn}^2 u\,du = \frac{E-k'^2K}{k^2K}u + \frac{\mathrm{Z}(u)}{k^2},$$

$$\int \mathrm{dn}^2 u\,du = \frac{E}{K}u + \mathrm{Z}(u),$$

$$\int \mathrm{ns}^2 u\,du = \frac{K-E}{K}u - \mathrm{Z}(u) - \frac{\mathrm{cs}\,u\,\mathrm{ds}\,u}{\mathrm{ns}\,u},$$

$$\int \mathrm{ds}^2 u\,du = -\frac{E-k'^2K}{K}u - \mathrm{Z}(u) - \frac{\mathrm{cs}\,u\,\mathrm{ds}\,u}{\mathrm{ns}\,u},$$

$$\int \mathrm{cs}^2 u\,du = -\frac{E}{K}u - \mathrm{Z}(u) - \frac{\mathrm{cs}\,u\,\mathrm{ds}\,u}{\mathrm{ns}\,u},$$

$$\int \mathrm{dc}^2 u\,du = \frac{K-E}{K}u - \mathrm{Z}(u) + \frac{\mathrm{dc}\,u\,\mathrm{sc}\,u}{\mathrm{nc}\,u},$$

$$\int \mathrm{nc}^2 u\,du = -\frac{E-k'^2K}{k'^2K}u - \frac{\mathrm{Z}(u)}{k'^2} + \frac{1}{k'^2}\frac{\mathrm{dc}\,u\,\mathrm{sc}\,u}{\mathrm{nc}\,u},$$

$$\int \mathrm{sc}^2 u\,du = -\frac{E}{k'^2K}u - \frac{\mathrm{Z}(u)}{k'^2} + \frac{1}{k'^2}\frac{\mathrm{dc}\,u\,\mathrm{sc}\,u}{\mathrm{nc}\,u},$$

$$\int \mathrm{cd}^2 u\,du = \frac{K-E}{k^2K}u - \frac{\mathrm{Z}(u)}{k^2} + \frac{\mathrm{sd}\,u\,\mathrm{cd}\,u}{\mathrm{nd}\,u},$$

$$\int \mathrm{sd}^2 u\,du = \frac{E-k'^2K}{k^2k'^2K}u + \frac{\mathrm{Z}(u)}{k^2k'^2} - \frac{1}{k'^2}\frac{\mathrm{sd}\,u\,\mathrm{cd}\,u}{\mathrm{nd}\,u},$$

$$\int \mathrm{nd}^2 u\,du = \frac{E}{k'^2K}u + \frac{\mathrm{Z}(u)}{k'^2} - \frac{k^2}{k'^2}\frac{\mathrm{sd}\,u\,\mathrm{cd}\,u}{\mathrm{nd}\,u}.$$

§ 27. By arranging the functions according to their numerators, and expressing the results in terms of $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, the system of formulæ assumes the following form:

$$\int \operatorname{sn}^2 u\, du = \frac{K-E}{k^2 K}\, u - \frac{Z(u)}{k^2},$$

$$\int \operatorname{sc}^2 u\, du = -\frac{E}{k'^2 K}\, u - \frac{Z(u)}{k'^2} + \frac{1}{k'^2}\,\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u},$$

$$\int \operatorname{sd}^2 u\, du = \frac{E-k'^2 K}{k^2 k'^2 K}\, u + \frac{Z(u)}{k^2 k'^2} - \frac{1}{k'^2}\,\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u},$$

$$\int \operatorname{cn}^2 u\, du = \frac{E-k'^2 K}{k^2 K}\, u + \frac{Z(u)}{k^2},$$

$$\int \operatorname{cs}^2 u\, du = -\frac{E}{K}\, u - Z(u) - \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u},$$

$$\int \operatorname{cd}^2 u\, du = \frac{K-E}{k^2 K}\, u - \frac{Z(u)}{k^2} + \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u},$$

$$\int \operatorname{dn}^2 u\, du = \frac{E}{K}\, u + Z(u),$$

$$\int \operatorname{ds}^2 u\, du = -\frac{E-k'^2 K}{K}\, u - Z(u) - \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u},$$

$$\int \operatorname{dc}^2 u\, du = \frac{K-E}{K}\, u - Z(u) + \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u},$$

$$\int \operatorname{ns}^2 u\, du = \frac{K-E}{K}\, u - Z(u) - \frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn} u},$$

$$\int \operatorname{nc}^2 u\, du = -\frac{E-k'^2 K}{k'^2 K}\, u - \frac{Z(u)}{k'^2} + \frac{1}{k'^2}\,\frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u},$$

$$\int \operatorname{nd}^2 u\, du = \frac{E}{k'^2 K}\, u + \frac{Z(u)}{k'^2} - \frac{k^2}{k'^2}\,\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}.$$

§ 28. Since the limit of $\dfrac{a^n-1}{n}$ when $n=0$ is $\log a$, we obtain, by putting $n=0$ in the formulæ of § 18, the system of equations

$$\frac{d^2}{du^2}\log \mathrm{sn}\, u = k^2\, \mathrm{sn}^2 u - \frac{1}{\mathrm{sn}^2 u} = k^2\, \mathrm{sn}^2 u - \mathrm{ns}^2 u,$$

$$\frac{d^2}{du^2}\log \mathrm{cn}\, u = -k^2\, \mathrm{cn}^2 u - \frac{k'^2}{\mathrm{cn}^2 u} = -k^2\, \mathrm{cn}^2 u - k'^2\, \mathrm{nc}^2 u,$$

$$\frac{d^2}{du^2}\log \mathrm{dn}\, u = -\,\mathrm{dn}^2 u + \frac{k'^2}{\mathrm{dn}^2 u} = -\,\mathrm{dn}^2 u + k'^2\, \mathrm{nd}^2 u,$$

$$\frac{d^2}{du^2}\log \mathrm{ns}\, u = \mathrm{ns}^2 u - \frac{k^2}{\mathrm{ns}^2 u} = \mathrm{ns}^2 u - k^2\, \mathrm{sn}^2 u,$$

$$\frac{d^2}{du^2}\log \mathrm{ds}\, u = \mathrm{ds}^2 u + \frac{k^2 k'^2}{\mathrm{ds}^2 u} = \mathrm{ds}^2 u + k^2 k'^2\, \mathrm{sd}^2 u,$$

$$\frac{d^2}{du^2}\log \mathrm{cs}\, u = \mathrm{cs}^2 u - \frac{k'^2}{\mathrm{cs}^2 u} = \mathrm{cs}^2 u - k'^2\, \mathrm{sc}^2 u,$$

$$\frac{d^2}{du^2}\log \mathrm{dc}\, u = \mathrm{dc}^2 u - \frac{k^2}{\mathrm{dc}^2 u} = \mathrm{dc}^2 u - k^2\, \mathrm{cd}^2 u,$$

$$\frac{d^2}{du^2}\log \mathrm{nc}\, u = k'^2\, \mathrm{nc}^2 u + \frac{k^2}{\mathrm{nc}^2 u} = k'^2\, \mathrm{nc}^2 u + k^2\, \mathrm{cn}^2 u,$$

$$\frac{d^2}{du^2}\log \mathrm{sc}\, u = k'^2\, \mathrm{sc}^2 u - \frac{1}{\mathrm{sc}^2 u} = k'^2\, \mathrm{sc}^2 u - \mathrm{cs}^2 u,$$

$$\frac{d^2}{du^2}\log \mathrm{cd}\, u = k^2\, \mathrm{cd}^2 u - \frac{1}{\mathrm{cd}^2 u} = k^2\, \mathrm{cd}^2 u - \mathrm{dc}^2 u,$$

$$\frac{d^2}{du^2}\log \mathrm{sd}\, u = -k^2 k'^2\, \mathrm{sd}^2 u - \frac{1}{\mathrm{sd}^2 u} = -k^2 k'^2\, \mathrm{sd}^2 u - \mathrm{ds}^2 u,$$

$$\frac{d^2}{du^2}\log \mathrm{nd}\, u = -k'^2\, \mathrm{nd}^2 u + \frac{1}{\mathrm{nd}^2 u} = -\,k'^2\, \mathrm{nd}^2 u + \mathrm{dn}^2 u.$$

§ 29. These formulæ, six of which are identical with the other six, enable us to pass from $\int \phi^2 u\, du$ to $\int \frac{du}{\phi^2 u}$ and *vice versâ*, where $\phi(u)$ denotes any one of the twelve functions.

The six formulæ may be written

$$k^2 \int \mathrm{sn}^2 u\,du - \int \mathrm{ns}^2 u\,du = \frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u} = \frac{\mathrm{cs}\,u\,\mathrm{ds}\,u}{\mathrm{ns}\,u},$$

$$k^2 \int \mathrm{cn}^2 u\,du + k'^2 \int \mathrm{nc}^2 u\,du = \frac{\mathrm{sn}\,u\,\mathrm{dn}\,u}{\mathrm{cn}\,u} = \frac{\mathrm{sc}\,u\,\mathrm{dc}\,u}{\mathrm{nc}\,u},$$

$$\int \mathrm{dn}^2 u\,du - k'^2 \int \mathrm{nd}^2 u\,du = k^2 \frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u} = k^2 \frac{\mathrm{sd}\,u\,\mathrm{cd}\,u}{\mathrm{nd}\,u},$$

$$k'^2 \int \mathrm{sc}^2 u\,du - \int \mathrm{cs}^2 u\,du = \frac{\mathrm{nc}\,u\,\mathrm{dc}\,u}{\mathrm{sc}\,u} = \frac{\mathrm{ns}\,u\,\mathrm{ds}\,u}{\mathrm{cs}\,u},$$

$$k^2 k'^2 \int \mathrm{sd}^2 u\,du + \int \mathrm{ds}^2 u\,du = -\frac{\mathrm{nd}\,u\,\mathrm{cd}\,u}{\mathrm{sd}\,u} = -\frac{\mathrm{ns}\,u\,\mathrm{cs}\,u}{\mathrm{ds}\,u},$$

$$\int \mathrm{dc}^2 u\,du - k^2 \int \mathrm{cd}^2 u\,du = k'^2 \frac{\mathrm{nc}\,u\,\mathrm{sc}\,u}{\mathrm{dc}\,u} = k'^2 \frac{\mathrm{nd}\,u\,\mathrm{sd}\,u}{\mathrm{cd}\,u}.$$

It is easy to verify that the values of $\int \mathrm{sn}^2 u\,du$, &c. in § 26 satisfy these relations.

Only one of these formulæ leads to a trigonometrical equation of similar form, viz. the fourth, which by putting $k = 0$ becomes

$$\int \tan^2 u\,du - \int \cot^2 u\,du = \frac{1}{\sin u \cos u}.$$

§ 30. In connexion with the formulæ in § 26 it may be observed that corresponding to the identical relations

$$\left.\begin{aligned} \mathrm{sn}^2 u + \mathrm{cn}^2 u &= 1 \\ \mathrm{dn}^2 u + k^2\,\mathrm{sn}^2 u &= 1 \\ \mathrm{dn}^2 u - k^2\,\mathrm{cn}^2 u &= k'^2 \end{aligned}\right\},$$

we have in the case of the other groups the relations

$$\left.\begin{aligned} \mathrm{ns}^2 u - \mathrm{ds}^2 u &= k^2 \\ \mathrm{ns}^2 u - \mathrm{cs}^2 u &= 1 \\ \mathrm{ds}^2 u - \mathrm{cs}^2 u &= k'^2 \end{aligned}\right\},$$

$$\left.\begin{aligned} \mathrm{dc}^2 u - k'^2\,\mathrm{nc}^2 u &= k^2 \\ \mathrm{dc}^2 u - k'^2\,\mathrm{sc}^2 u &= 1 \\ \mathrm{nc}^2 u - \mathrm{sc}^2 u &= 1 \end{aligned}\right\},$$

$$\left.\begin{aligned} \mathrm{cd}^2 u + k'^2\,\mathrm{sd}^2 u &= 1 \\ k^2\,\mathrm{cd}^2 u + k'^2\,\mathrm{nd}^2 u &= 1 \\ \mathrm{nd}^2 u - k^2\,\mathrm{sd}^2 u &= 1 \end{aligned}\right\}.$$

By means of these relations we may deduce from any equation in § 26 the other two equations of the group to which it belongs.

Formulæ of reduction for $\int \mathrm{sn}^3 u\,du$, &c. *and* $\int \mathrm{sn}^4 u\,du$, &c., §§ 31, 32.

§ 31. By putting $n=1$ in the formulæ of reduction of § 21 we obtain the following equations which express the values of $\int \mathrm{sn}^3 u\,du$, &c., in terms of $\int \mathrm{sn}\,u\,du$, &c.,

$$\int \mathrm{sn}^3 u\,du = \frac{1+k^2}{2k^2}\int \mathrm{sn}\,u\,du + \frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{2k^2},$$

$$\int \mathrm{cn}^3 u\,du = \frac{k^2-k'^2}{2k^2}\int \mathrm{cn}\,u\,du + \frac{\mathrm{dn}\,u\,\mathrm{sn}\,u}{2k^2},$$

$$\int \mathrm{dn}^3 u\,du = \frac{1+k'^2}{2}\int \mathrm{dn}\,u\,du + \frac{k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u}{2},$$

$$\int \mathrm{ns}^3 u\,du = \frac{1+k^2}{2}\int \mathrm{ns}\,u\,du - \frac{\mathrm{ds}\,u\,\mathrm{cs}\,u}{2},$$

$$\int \mathrm{ds}^3 u\,du = -\frac{k^2-k'^2}{2}\int \mathrm{ds}\,u\,du - \frac{\mathrm{cs}\,u\,\mathrm{ns}\,u}{2},$$

$$\int \mathrm{cs}^3 u\,du = -\frac{1+k'^2}{2}\int \mathrm{cs}\,u\,du - \frac{\mathrm{ns}\,u\,\mathrm{ds}\,u}{2},$$

$$\int \mathrm{dc}^3 u\,du = \frac{1+k^2}{2}\int \mathrm{dc}\,u\,du + \frac{k'^2\,\mathrm{nc}\,u\,\mathrm{sc}\,u}{2},$$

$$\int \mathrm{nc}^3 u\,du = -\frac{k^2-k'^2}{2k'^2}\int \mathrm{nc}\,u\,du + \frac{\mathrm{sc}\,u\,\mathrm{dc}\,u}{2k'^2},$$

$$\int \mathrm{sc}^3 u\,du = -\frac{1+k'^2}{2k'^2}\int \mathrm{sc}\,u\,du + \frac{\mathrm{dc}\,u\,\mathrm{nc}\,u}{2k'^2},$$

$$\int \mathrm{cd}^3 u\,du = \frac{1+k^2}{2k^2}\int \mathrm{cd}\,u\,du - \frac{k'^2\,\mathrm{sd}\,u\,\mathrm{nd}\,u}{2k^2},$$

$$\int \mathrm{sd}^3 u\,du = \frac{k^2-k'^2}{2k^2k'^2}\int \mathrm{sd}\,u\,du - \frac{\mathrm{nd}\,u\,\mathrm{cd}\,u}{2k^2k'^2},$$

$$\int \mathrm{nd}^3 u\,du = \frac{1+k'^2}{2k'^2}\int \mathrm{nd}\,u\,du - \frac{k^2\,\mathrm{cd}\,u\,\mathrm{sd}\,u}{2k'^2}.$$

§ 32. Similarly, by putting $n=2$ in the formulæ of § 21, we obtain the following values of $\int \mathrm{sn}^4 u\,du$, &c.

$$\int \mathrm{sn}^4 u\,du = \frac{2(1+k^2)}{3k^2}\int \mathrm{sn}^2 u\,du + \frac{\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{3k^2} - \frac{u}{3k^2},$$

$$\int \mathrm{cn}^4 u\,du = \frac{2(k^2-k'^2)}{3k^2}\int \mathrm{cn}^2 u\,du + \frac{\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{3k^2} + \frac{k'^2 u}{3k^2},$$

$$\int \mathrm{dn}^4 u\,du = \frac{2(1+k'^2)}{3}\int \mathrm{dn}^2 u\,du + \frac{k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{3} - \frac{k'^2 u}{3},$$

$$\int \mathrm{ns}^4 u\,du = \frac{2(1+k^2)}{3}\int \mathrm{ns}^2 u\,du - \frac{\mathrm{ns}\,u\,\mathrm{ds}\,u\,\mathrm{cs}\,u}{3} - \frac{k^2 u}{3},$$

$$\int \mathrm{ds}^4 u\,du = -\frac{2(k^2-k'^2)}{3}\int \mathrm{ds}^2 u\,du - \frac{\mathrm{ns}\,u\,\mathrm{ds}\,u\,\mathrm{cs}\,u}{3} + \frac{k^2k'^2 u}{3},$$

$$\int \mathrm{cs}^4 u\,du = -\frac{2(1+k'^2)}{3}\int \mathrm{cs}^2 u\,du - \frac{\mathrm{ns}\,u\,\mathrm{ds}\,u\,\mathrm{cs}\,u}{3} - \frac{k'^2 u}{3},$$

$$\int \mathrm{dc}^4 u\,du = \frac{2(1+k^2)}{3}\int \mathrm{dc}^2 u\,du + \frac{k'^2\,\mathrm{dc}\,u\,\mathrm{nc}\,u\,\mathrm{sc}\,u}{3} - \frac{k^2 u}{3},$$

$$\int \mathrm{nc}^4 u\,du = -\frac{2(k^2-k'^2)}{3k'^2}\int \mathrm{nc}^2 u\,du + \frac{\mathrm{dc}\,u\,\mathrm{nc}\,u\,\mathrm{sc}\,u}{3k'^2} + \frac{k^2 u}{3k'^2},$$

$$\int \mathrm{sc}^4 u\,du = -\frac{2(1+k'^2)}{3k'^2}\int \mathrm{sc}^2 u\,du + \frac{\mathrm{dc}\,u\,\mathrm{nc}\,u\,\mathrm{sc}\,u}{3k'^2} - \frac{u}{3k'^2},$$

$$\int \mathrm{cd}^4 u\,du = \frac{2(1+k^2)}{3k^2}\int \mathrm{cd}^2 u\,du - \frac{k'^2\,\mathrm{cd}\,u\,\mathrm{sd}\,u\,\mathrm{nd}\,u}{3k^2} - \frac{u}{3k^2},$$

$$\int \mathrm{sd}^4 u\,du = \frac{2(k^2-k'^2)}{3k^2k'^2}\int \mathrm{sd}^2 u\,du - \frac{\mathrm{cd}\,u\,\mathrm{sd}\,u\,\mathrm{nd}\,u}{3k^2k'^2} + \frac{u}{3k^2k'^2},$$

$$\int \mathrm{nd}^4 u\,du = \frac{2(1+k'^2)}{3k'^2}\int \mathrm{nd}^2 u\,du - \frac{k^2\,\mathrm{cd}\,u\,\mathrm{sd}\,u\,\mathrm{nd}\,u}{3k'^2} - \frac{u}{3k'^2}.$$

Formulæ for $\int \mathrm{sn}^5 u\,du, \ldots \int \mathrm{sn}^9 u\,du$, *&c. in terms of* $\int \mathrm{sn}\,u\,du$, *&c.*, §§ 33, 34.

§ 33. All the formulæ of reduction in § 22 are included in the form

$$(n+1)\,u_{n+2} - n\,(a+b)\,u_n + (n-1)\,ab\,u_{n-2} - h_n = 0,$$

where, for example, if $u_n = \int (k\,\mathrm{sn}\,u)^n\,du$, then $a=1$, $b=k^2$ and $h_n = k^n\,\mathrm{sn}^{n-1}u\,\mathrm{cn}\,u\,\mathrm{dn}\,u$; if $u_n = \int (k\,\mathrm{cn}\,u)^n\,du$, then $a=k^2$, $b=-k'^2$, $h_n = k^n\,\mathrm{cn}^{n-1}u\,\mathrm{dn}\,u\,\mathrm{sn}\,u$, &c.

Putting $n = 1, 3, 5, \ldots$ we have

$$\begin{aligned}
2u_3 &= (a+b)\,u_1 + h_1,\\
4u_5 &= 3(a+b)\,u_3 - 2abu_1 + h_3,\\
6u_7 &= 5(a+b)\,u_5 - 4abu_3 + h_5,\\
8u_9 &= 7(a+b)\,u_7 - 6abu_5 + h_7, \text{ \&c.},
\end{aligned}$$

and from these equations, we find

$$\begin{aligned}
u_3 &= U_1^{(3)}u_1 + H_1^{(3)}h_1,\\
u_5 &= U_1^{(5)}u_1 + H_1^{(5)}h_1 + H_3^{(5)}h_3,\\
u_7 &= U_1^{(7)}u_1 + H_1^{(7)}h_1 + H_3^{(7)}h_3 + H_5^{(7)}h_5,\\
u_9 &= U_1^{(9)}u_1 + H_1^{(9)}h_1 + H_3^{(9)}h_3 + H_5^{(9)}h_5 + H_7^{(9)}h_7, \text{ \&c.}
\end{aligned}$$

The values of the U- and H-coefficients up to the ninth order are given by the equations

$$\begin{aligned}
U_1^{(3)} &= \tfrac{1}{2}\ (1, 1\|a, b),\\
U_1^{(5)} &= \tfrac{1}{8}\ (3, 2, 3\|a, b)^2,\\
U_1^{(7)} &= \tfrac{1}{16}\ (5, 3, 3, 5\|a, b)^3,\\
U_1^{(9)} &= \tfrac{1}{128}\ (35, 20, 18, 20, 35\|a, b)^4.
\end{aligned}$$

$$\begin{aligned}
H_1^{(3)} &= \tfrac{1}{2},\\
H_1^{(5)} &= \tfrac{3}{8}\ (1, 1\|a, b),\\
H_1^{(7)} &= \tfrac{1}{48}\ (15, 14, 15\|a, b)^2,\\
H_1^{(9)} &= \tfrac{5}{384}\ (21, 19, 19, 21\|a, b)^3,\\
H_3^{(5)} &= \tfrac{1}{4},\\
H_3^{(7)} &= \tfrac{5}{24}\ (1, 1\|a, b),\\
H_3^{(9)} &= \tfrac{1}{192}\ (35, 34, 35\|a, b)^2,\\
H_5^{(7)} &= \tfrac{1}{6},\\
H_5^{(9)} &= \tfrac{7}{48}\ (1, 1\|a, b),\\
H_7^{(9)} &= \tfrac{1}{8},
\end{aligned}$$

where the notation $(A_1, A_2, \ldots A_{n+1}\|a, b)^n$ denotes as usual $A_1a^n + A_2a^{n-1}b \ldots + A_{n+1}b^n$.

§ 34. The values of u_n, a, b, h_n for the different functions are shown below

u_n	a,	b,	h_n,
$\int(k\,\mathrm{sn}\,u)^n\,du$,	1,	k^2,	$k^n\ \mathrm{sn}^{n-1}u\,\mathrm{cn}\,u\,\mathrm{dn}\,u$,
$\int(\mathrm{ns}\,u)^n\,du$,	1,	k^2,	$-\ \mathrm{ns}^{n-1}u\,\mathrm{ds}\,u\,\mathrm{cs}\,u$,
$\int(\mathrm{dc}\,u)^n\,du$,	1,	k^2,	$k'^2\,\mathrm{dc}^{n-1}u\,\mathrm{nc}\,u\,\mathrm{sc}\,u$,
$\int(k\,\mathrm{cd}\,u)^n\,du$,	1,	k^2,	$-k^nk'^2\,\mathrm{cd}^{n-1}u\,\mathrm{sd}\,u\,\mathrm{nd}\,u$,
$\int(k\,\mathrm{cn}\,u)^n\,du$,	k^2,	$-k'^2$,	$k^n\ \mathrm{cn}^{n-1}u\,\mathrm{dn}\,u\,\mathrm{sn}\,u$,
$\int(\mathrm{ds}\,u)^n\,du$,	k'^2,	$-k^2$,	$-\ \mathrm{ds}^{n-1}u\,\mathrm{cs}\,u\,\mathrm{ns}\,u$,
$\int(k'\,\mathrm{nc}\,u)^n\,du$,	k'^2,	$-k^2$,	$k'^n\,\mathrm{nc}^{n-1}u\,\mathrm{sc}\,u\,\mathrm{dc}\,u$,
$\int(kk'\,\mathrm{sd}\,u)^n\,du$,	k^2,	$-k'^2$,	$-k^nk'^n\,\mathrm{sd}^{n-1}u\,\mathrm{nd}\,u\,\mathrm{cd}\,u$,
$\int(\mathrm{dn}\,u)^n\,du$,	1,	k'^2,	$k^2\ \mathrm{dn}^{n-1}u\,\mathrm{sn}\,u\,\mathrm{cn}\,u$,
$\int(\mathrm{cs}\,u)^n\,du$,	-1,	$-k'^2$,	$-\ \mathrm{cs}^{n-1}u\,\mathrm{ns}\,u\,\mathrm{ds}\,u$,
$\int(k'\,\mathrm{sc}\,u)^n\,du$,	-1,	$-k'^2$,	$k'^n\,\mathrm{sc}^{n-1}u\,\mathrm{dc}\,u\,\mathrm{nc}\,u$,
$\int(k'\,\mathrm{nd}\,u)^n\,du$,	1,	k'^2,	$-k^2k'^n\,\mathrm{nd}^{n-1}u\,\mathrm{cd}\,u\,\mathrm{sd}\,u$.

Thus, for example,

$$\int(k\,\mathrm{sn}\,u)^6\,du$$
$$= \tfrac{1}{8}\ (3, 2, 3\text{X}1, k^2)^2 \int(k\,\mathrm{sn}\,u)\,du$$
$$+ \tfrac{3}{8}\ (1, 1\text{X}1, k^2).k\,\mathrm{cn}\,u\,\mathrm{dn}\,u$$
$$+ \tfrac{1}{4}.k^3\,\mathrm{sn}^2u\,\mathrm{cn}\,u\,\mathrm{dn}\,u,$$

$$\int(kk'\,\mathrm{sd}\,u)^9\,du$$
$$= {}_1\tfrac{1}{2}{}_8\,(35, 20, 18, 20, 25\text{X}k^2, -k'^2)^4 \int(kk'\,\mathrm{sd}\,u)\,du$$
$$+ {}_3\tfrac{5}{8}{}_4\,(21, 19, 19, 21\text{X}k^2, -k'^2)^3.-kk'\qquad \mathrm{nd}\,u\,\mathrm{cd}\,u$$
$$+ {}_1\tfrac{1}{9}{}_2\,(35, 34, 35\text{X}k^2, -k'^2)^2\quad .-k^3k'^3\,\mathrm{sd}^2u\,\mathrm{nd}\,u\,\mathrm{cd}\,u$$
$$+ \tfrac{7}{4}{}_8\,(1, 1\text{X}k^2, -k'^2)\qquad .-k^5k'^5\,\mathrm{sd}^4u\,\mathrm{nd}\,u\,\mathrm{cd}\,u$$
$$+ \tfrac{1}{8}\qquad .-k^7k'^7\,\mathrm{sd}^6u\,\mathrm{nd}\,u\,\mathrm{cd}\,u.$$

Formulæ for $\int\mathrm{sn}^6u\,du, \ldots \int\mathrm{sn}^{10}u\,du$, *&c. in terms of* $\int\mathrm{sn}^2u$, *&c.*, §§ 35, 36.

§ 35. Putting $n=2, 4, 6, \ldots$ in the formula at the beginning of of § 33, viz.

$$(n+1)\,u_{n+2} - n\,(a+b)\,u_n + (n-1)\,abu_{n-2} - h_n = 0,$$

we have

$$3u_4 = 2(a+b)u_2 - abu_0 + h_2,$$
$$5u_6 = 4(a+b)u_4 - 3abu_2 + h_4,$$
$$7u_8 = 6(a+b)u_6 - 5abu_4 + h_6,$$
$$9u_{10} = 8(a+b)u_8 - 7abu_6 + h_8, \quad \&c.,$$

whence

$$u_4 = U_2^{(4)}u_2 - U_0^{(4)}u_0 + H_2^{(4)}h_2,$$
$$u_6 = U_2^{(6)}u_2 - U_0^{(6)}u_0 + H_2^{(6)}h_2 + H_4^{(6)}h_4,$$
$$u_8 = U_2^{(8)}u_2 - U_0^{(8)}u_0 + H_2^{(8)}h_2 + H_4^{(8)}h_4 + H_6^{(8)}h_6,$$
$$u_{10} = U_2^{(10)}u_2 - U_0^{(10)}u_0 + H_2^{(10)}h_2 + H_4^{(10)}h_4 + H_6^{(10)}h_6 + H_8^{(10)}h_8,$$
$$\&c. = \&c.$$

The values of the U- and H- coefficients up to the tenth order are given by the equations

$$U_2^{(4)} = \tfrac{2}{3}\ (1, 1\nabla a, b),$$
$$U_2^{(6)} = \tfrac{1}{15}\ (8, 7, 8\nabla a, b)^2,$$
$$U_2^{(8)} = \tfrac{8}{105}\,(6, 5, 5, 6\nabla a, b)^3,$$
$$U_2^{(10)} = \tfrac{1}{315}\,(128, 104, 99, 104, 128\nabla a, b)^4,$$
$$U_0^{(4)} = \tfrac{1}{3}\ ab,$$
$$U_0^{(6)} = \tfrac{4}{15}\ ab\,(1, 1\nabla a, b),$$
$$U_0^{(8)} = \tfrac{1}{105}\,ab\,(24, 23, 24\nabla a, b)^2,$$
$$U_0^{(10)} = \tfrac{4}{315}\,ab\,(16, 15, 15, 16\nabla a, b)^3,$$

$$H_2^{(4)} = \tfrac{1}{3},$$
$$H_2^{(6)} = \tfrac{4}{15}\ (1, 1\nabla a, b),$$
$$H_2^{(8)} = \tfrac{1}{105}\,(24, 23, 24\nabla a, b)^2,$$
$$H_2^{(10)} = \tfrac{4}{315}\,(16, 15, 15, 16\nabla a, b)^3,$$
$$H_4^{(6)} = \tfrac{1}{5},$$
$$H_4^{(8)} = \tfrac{6}{35}\ (1, 1\nabla a, b),$$
$$H_4^{(10)} = \tfrac{1}{315}\,(48, 47, 48\nabla a, b)^2,$$
$$H_6^{(8)} = \tfrac{1}{7},$$
$$H_6^{(10)} = \tfrac{8}{63}\ (1, 1\nabla a, b),$$
$$H_8^{(10)} = \tfrac{1}{9}.$$

The values of u_n, a, b, h_n were given in § 34; and we have also $u_0 = u$. It is evident that $U_0^{(2n)} = abH_0^{(2n)}$.

Thus, for example,

$$\begin{aligned}\int(\mathrm{cs}\,u)^6\,du = {} & \tfrac{1}{15}\,(8, 7, 8\text{\textbackslash\textbackslash}1, k'^2)^2 \int(\mathrm{cs}\,u)^2\,du \\ & + \tfrac{4}{15}\,k'^2\,(1, 1\text{\textbackslash\textbackslash}1, k'^2)\,u \\ & + \tfrac{4}{15}\,(1, 1\text{\textbackslash\textbackslash}1, k'^2)\,\mathrm{cs}\,u\,\mathrm{ns}\,u\,\mathrm{ds}\,u \\ & - \tfrac{1}{6}\,\mathrm{cs}^3 u\,\mathrm{ns}\,u\,\mathrm{ds}\,u.\end{aligned}$$

§ 36. It may be remarked that the coefficients with the same suffix may be calculated independently; thus, for example, we have

$$\begin{aligned} 6H_1^{(7)} &= 5\,(a+b)\,H_1^{(5)} - 4abH_1^{(3)}, \\ 8H_1^{(9)} &= 7\,(a+b)\,H_1^{(7)} - 6abH_1^{(5)}, \\ \&\mathrm{c.} &= \quad \&\mathrm{c.}, \\ 7H_2^{(8)} &= 6\,(a+b)\,H_2^{(6)} - 5abH_2^{(4)}, \\ \&\mathrm{c.} &= \quad \&\mathrm{c.} \end{aligned}$$

ON THE DERIVATION OF ELLIPTIC FUNCTION FORMULÆ BY TRANSFORMATION TO THE RECIPROCAL AND TO THE COMPLEMENTARY MODULUS.

By *William Woolsey Johnson*, Professor of Mathematics in the Naval Academy, Annapolis, Maryland, U.S.A.

1. When in a formula the modulus is arbitrary, and the arguments are all homogeneous functions of arbitrary arguments, we are at liberty to multiply all the arguments by the same factor, and to make any change in the modulus. Hence, since

$$\mathrm{sn}\,(iu, k') = \frac{i\,\mathrm{sn}\,u}{\mathrm{cn}\,u}, \quad \text{and} \quad \mathrm{sn}\left(ku, \frac{1}{k}\right) = k\,\mathrm{sn}\,u,$$

$$\mathrm{cn}\,(iu, k') = \frac{1}{\mathrm{cn}\,u}, \qquad \mathrm{cn}\left(ku, \frac{1}{k}\right) = \mathrm{dn}\,u,$$

$$\mathrm{dn}\,(iu, k') = \frac{\mathrm{dn}\,u}{\mathrm{cn}\,u}, \qquad \mathrm{dn}\left(ku, \frac{1}{k}\right) = \mathrm{cn}\,u,$$

we may, adopting Mr. Glaisher's notation (compare the table on page 120 of this volume, *Messenger*, Dec. 1881), in any such formula replace the quantities k and k', and each sn, cn and dn, by the quantity opposite to it in either of the schemes

k	k'	k	$\frac{1}{k}$
k'	k	k'	$\frac{ik'}{k}$
sn	i sc	sn	k sn
cn	nc	cn	dn
dn	dc	dn	cn.

2. If, as suggested by the notation, we substitute $\frac{s}{n}$, $\frac{c}{n}$ and $\frac{d}{n}$ for sn, cn and dn, and render the formula homogeneous by multiplying by the proper power of each n, the schemes may be written as follows

k	k'	k	$\frac{1}{k}$
k'	k	k'	$\frac{ik'}{k}$
s	is	s	ks
c	n	c	d
d	d	d	c
n	c	n	n.

3. For example, using the subscripts 1 and 2 to denote the arguments u and v, as in the system of values of the functions of $u+v$, *Messenger*, vol. x., p. 106, Nov. 1880, and denoting the numerators in the first column by S, C and D, and the common denominator by N, we may write

$$S=\frac{1}{n_1^2n_2^2}(s_1n_1c_2d_2+s_2n_2c_1d_1),$$

$$C=\frac{1}{n_1^2n_2^2}(c_1c_2n_1n_2-s_1s_2d_1d_2),$$

$$D=\frac{1}{n_1^2n_2^2}(d_1d_2n_1n_2-k^2s_1s_2c_1c_2),$$

$$N=\frac{1}{n_1^2n_2^2}(n_1^2n_2^2-k^2s_1^2s_2^2).$$

Applying the first of the above transformations, we have

$$iS' = \frac{i}{c_1^2 c_2^2}(s_1 c_1 d_2 n_2 + s_2 c_2 d_1 n_1),$$

$$N' = \frac{1}{c_1^2 c_2^2}(c_1 c_2 n_1 n_2 + s_1 s_2 d_1 d_2),$$

$$D' = \frac{1}{c_1^2 c_2^2}(c_1 c_2 d_1 d_2 + k'^2 s_1 s_2 n_1 n_2),$$

$$C' = \frac{1}{c_1^2 c_2^2}(c_1^2 c_2^2 \quad + k'^2 s_1^2 s_2^2),$$

and the ratios of S', C', D' and N' give the third set of values of the functions of $u+v$ in the table quoted above.

4. It is obvious that two successive applications of the transformation reproduce in all cases the original formula, except that the sign of s is changed, which is equivalent to changing the signs of all the arguments. In like manner a double application of the second transformation reproduces the original formula except that the sign of k' is changed. New formulæ may, however, be produced by alternate applications of the two transformations.

Only five new formulæ can ever be derived from a given formula in this way, the quantities which successively take the place of the original k being

$$k', \quad \frac{1}{k'}, \quad \frac{ik}{k'}, \quad \frac{ik'}{k}, \quad \frac{1}{k},$$

and k being reproduced at the sixth step. The new formulæ may be found at once by transforming the original formula to these five values of the modulus. If we denote the transformation to the complementary modulus by C, and that to the reciprocal modulus by R, it is plain that the new formulæ may be derived by the transformations RC, CR and CRC (which last is equivalent to RCR). The complete set of schemes is as follows

C	
k	k'
k'	k
s	is
c	n
d	d
n	c

R	
k	$\frac{1}{k}$
k'	$\frac{ik'}{k}$
s	ks
c	d
d	c
n	n

RC	
k	$\frac{ik'}{k}$
k'	$\frac{1}{k}$
s	iks
c	n
d	c
n	d.

	CR
k	$\frac{1}{k'}$
k'	$\frac{ik}{k'}$
s	$ik's$
c	d
d	n
n	c

	CRC
k	$\frac{ik}{k'}$
k'	$\frac{1}{k'}$
s	$k's$
c	c
d	n
n	$d.$

If, for example, we apply these five transformations to the standard formula for cn $(u+v)$, written in the form

$$\text{cn}\,(u+v)=\frac{c_1c_2n_1n_2-s_1s_2d_1d_2}{n_1^{\,2}n_2^{\,2}-k^2s_1^{\,2}s_2^{\,2}},$$

we obtain

$$\text{nc}\,(u+v)=\frac{c_1c_2n_1n_2+s_1s_2d_1d_2}{c_1^{\,2}c_2^{\,2}-k'^2s_1^{\,2}s_2^{\,2}},$$

$$\text{dn}\,(u+v)=\frac{d_1d_2n_1n_2-k^2s_1s_2c_1c_2}{n_1^{\,2}n_2^{\,2}-k^2s_1^{\,2}s_2^{\,2}},$$

$$\text{nd}\,(u+v)=\frac{d_1d_2n_1n_2+k^2s_1s_2c_1c_2}{d_1^{\,2}d_2^{\,2}+k^2k'^2s_1^{\,2}s_2^{\,2}},$$

$$\text{dc}\,(u+v)=\frac{c_1c_2d_1d_2+k'^2s_1s_2n_1n_2}{c_1^{\,2}c_2^{\,2}-k'^2s_1^{\,2}s^2},$$

$$\text{cd}\,(u+v)=\frac{c_1c_2d_1d_2-k'^2s_1s_2n_1n_2}{d_1^{\,2}d_2^{\,2}+k^2k'^2s_1^{\,2}s_2^{\,2}}.$$

Replacing n by unity and substituting for the d's and c's in the denominators their values in terms of s's, these six formulæ are equivalent to the values of cn $(u+v)$ and dn $(u+v)$ given in the second, fourth and fifth columns of the table on p. 106, *Messenger*, Nov. 1880.

Of course it frequently happens that one of the transformations R or C reproduces the original formula. When this is the case only two transformations exist; thus, if R reproduces the original formula it is plain that CR gives the same result as C, and RCR the same result as RC.

REDUCTION OF $\int \frac{dx}{(1-x^3)^{\frac{2}{3}}}$ TO ELLIPTIC INTEGRALS.

By Professor *Cayley*.

WRITING s, c, d for the sn, cn, and dn of u to a modulus k, which will be determined, and denoting by θ a constant which will also be determined, the formula of reduction is

$$x = \frac{-1 + \theta scd}{1 + \theta scd},$$

To find from this the value of $y, = \sqrt[3]{(1-x^3)}$, putting for shortness $X = \theta scd$, the formula is $x = \frac{-1+X}{1+X}$, and we thence have

$$y^3, = 1 - x^3, = \frac{2(1+3X^2)}{(1+X)^3},$$

where

$$1 + 3X^2 = 1 + 3\theta^2 s^2 (1 - s^2)(1 - k^2 s^2)$$

$$= 1 + 3\theta^2 s^2 - 3\theta^2 (1 + k^2) s^4 + 3\theta^2 k^2 s^6$$

may be put equal to $(1 + \theta^2 s^2)^3$, that is

$$= 1 + 3\theta^2 s^2 + 3\theta^4 s^4 \qquad + \theta^6 s^6;$$

viz. this will be the case if

$$3\theta^4 = -3\theta^2 (1 + k^2), \quad \theta^6 = 3\theta^2 k^2;$$

that is $\theta^2 = -1 - k^2, \qquad \theta^4 = 3k^2;$

these give $k^4 - k^2 + 1 = 0;$

that is $k^2 = \omega$, if $\omega = -\frac{1}{2} + \frac{1}{2} i\sqrt{3}$, an imaginary cube root of unity; and then

$$\theta^2 = -1 + \omega, \quad = \omega^2 (\omega^2 - \omega), \quad = -i\omega^2 \sqrt{3};$$

that is $\theta = \pm \frac{(1-\sqrt{3}) - i(1+\sqrt{3})}{2\sqrt{2}} \sqrt[4]{3},$

as may be verified by squaring.

Hence finally, θ and k denoting the values just obtained,

$$x = \frac{-1 + \theta scd}{1 + \theta scd}.$$

$$y = \sqrt{(1 - x^3)} = \frac{\sqrt[3]{2}\,(1 + \theta^2 s^2)}{1 + \theta scd};$$

or, writing as before, $X = \theta scd$, we have

$$dx = \frac{2dX}{(1+X)^2}, \quad y^2 = \frac{2^{\frac{2}{3}}(1 + \theta^2 s^2)}{(1+X)^2};$$

whence $$\frac{dx}{(1-x^3)^{\frac{2}{3}}}, \quad = \frac{dx}{y^2}, \quad = \frac{2^{\frac{1}{3}} dX}{(1 + \theta^2 s^2)^2},$$

and then

$$dX = \theta\{1 - 2(1 + k^2)s^3 + 3k^2 s^4\}\,du, \quad = \theta(1 + \theta^2 s^2)^2\,du;$$

that is $$\frac{dx}{(1-x^3)^{\frac{2}{3}}} = 2^{\frac{1}{3}}\theta . du;$$

or say $$\int \frac{dx}{(1-x^3)^{\frac{2}{3}}} = 2^{\frac{1}{3}}\theta u,$$

the required formula.

SOME APPLICATIONS OF SYMBOLIC METHODS.

By *Arthur Buchheim*, *B.A.*, Scholar of New College, Oxford.

THE following paper contains proofs by symbolic methods (1) of the formula for the n^{th} differential coefficient of a function, (2) of Euler's theorems on homogeneous functions, (3) of Staudt's theorem relating to the numbers of Bernoulli.

1. Let

$$y = y(x), \; z = z(x) = z(y), \; D_1 = \frac{d}{dx}, \; D_2 = \frac{d}{dy},$$

then $z(x+h) = e^{hD_1} z = e^{kD_2} z$, where $k = y(x+h) - y(x)$;

therefore $\frac{D_1^n}{n!}$ = coefficient of h^n in $e^{\{y(x+h)-y(x)\}D_2}$.

Now $$e^{\{y(x+h)-y(x)\}D_2} = \sum \frac{\{y(x+h) - y(x)\}^m}{m!} D_2^m.$$

Therefore $$\frac{D_1^n}{n!} = \sum_m \frac{U_m D_2^m}{m!},$$

where U_m denotes the coefficient of h^n in $\{y(x+h) - y(x)\}^m$, viz. it is the value of $D_h^n \frac{\{y(x+h) - y(x)\}^m}{n!}$ for $h = 0$.

This is the formula given, as due to R. Hoppe and U. Meyer, in Schlömilch's *Compendium der höheren Analysis.*

We have $y(x+h) - y(x) = hy'(x) + \frac{h^2}{2!}y''(x) + \dots$,

and we hence obtain the formula given by M. Faà de Bruno, viz.

$$\frac{d^n z}{dx^n} = \sum \frac{n!}{k_1!\, k_2! \dots k_n!} \frac{d^\rho z}{dy^\rho} \left(\frac{y'}{1}\right)^{k_1} \left(\frac{y''}{2!}\right)^{k_2} \dots \left(\frac{y^{(n)}}{n!}\right)^{k_n},$$

where the summation extends to positive integral values of $\rho, k_1, k_2, \dots k_n$, satisfying the conditions

$$\rho = k_1 + k_2 + \dots + k_n,$$
$$n = k_1 + 2k_2 + \dots + nk_n.$$

The formula follows at once by an application of the multinomial theorem.

2. Let ϕ be homogeneous and of the n^{th} degree in $x_1, x_2, \dots x_r$, then it can be shown in various ways that

$$\left(x_1 \frac{d}{dx_1} + x_2 \frac{d}{dx_2} + \dots\right) \phi = n\phi;$$

now, adopting a notation of Prof. Cayley's, write x for the set $x_1 \dots x_n$, and D for the set $\frac{d}{dx_1} \dots \frac{d}{dx_n}$, so that $\Sigma x_i \frac{d}{dx_i}$ is written xD.* I use xD' to denote the operation when it operates on the variables in xD as well as on those in ϕ, and xD'' to denote the operation when it operates only on the variables in xD.

Assume $(xD)^m \phi = P = n(n-1)\dots(n-m+1)\phi$.

Then $nP = (xD')(xD)^m \phi = (xD)^{m+1}\phi + (xD'')(xD)^m \phi$.

In $(xD'')(xD)^m\phi$ the operator D can be treated as constant, and we therefore have

$$(xD'')(xD)^m\phi = m(xD)(xD)^{m-1}\phi = m(xD)^m\phi = mP.$$

Therefore $(xD)^{m+1}\phi = (n-m)P = n(n-1)\dots(n-m)\phi$.

This proof seems to show the true nature of the theorems more clearly than the ordinary proof by means of Taylor's theorem.

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* To the extent to which the notation is used above it is of course practically an adoption of Grassmann's methods.

3. The proof of Staudt's theorem given by Schläfli (*Quarterly Journal*, VI. 75), can be simplified by the use of symbolic methods.

Writing $\frac{1}{e^x-1}=\frac{1}{x}-\frac{1}{2}+\frac{B_1}{2!}x\ldots(-)^{n-1}\frac{B_{2n-1}}{2n!}x^{2n-1}\ldots,$

we have

$$(-)^{q-1}B_{2q-1}=\frac{\log(1+\Delta)0^{2q}}{\Delta}=\sum\frac{(-)^n n!}{n+1}\frac{\Delta^n 0^{2q}}{n!},$$

and $\frac{\Delta^n 0^{2q}}{n!}$ is always an integer, for we have

$$\frac{\Delta^n 0^m}{n!}=\frac{\Delta^{n-1}0^{m-1}}{n-1!}+n\frac{\Delta^n 0^{m-1}}{n!}.$$

Obviously if $n+1$ is the product of two unequal factors, they will both occur as factors in $n!$; and therefore $\frac{n!}{n+1}$ is an integer; this is also true if $n+1$ is a square $=\rho^2$; for if $\rho>2$, ρ and 2ρ both occur as factors in $n!$ and $\frac{n!}{n+1}$ is an integer; but for $\rho=2$, which gives $n=3$, we have $\Delta^3 0^{2q}=3^{2q}-3.2^{2q}+3.1^{2q}$, which is divisible by 4. Thus if $n+1$ is not a prime the term $\frac{\Delta^n 0^{2q}}{n+1}$ is always an integer.

Now let $n+1$ be a prime, let $2q\equiv u\,(\mathrm{mod}.n)$, then

$$\Delta^n 0^{2q}=(E-1)^n 0^{2q}\equiv\Delta^n 0^u\equiv 0\ (\mathrm{mod}.n+1),\ \text{if } u<n;$$

but if $u=n$, $\Delta^n 0^{2q}\equiv\Delta^n 0^n\equiv n!\equiv -1\,(\mathrm{mod}.\,n+1)$.

Thus $\frac{\Delta^n 0^{2q}}{n+1}$ is an integer unless $2q$ is divisible by n, and the term corresponding to $n+1=2$ is $-\frac{1}{2}$, and thus we have Staudt's theorem

$$B_{2q-1}=\text{integer}+(-1)^q\left(\tfrac{1}{2}+\sum\frac{1}{2m+1}\right),$$

where the summation extends to those divisors, m, of q for which $2m+1$ is prime.

STRESS AND STRAIN IN CYLINDRICAL AND POLAR COORDINATES.

By *R. R. Webb, M.A.*

Strain in cylindrical coordinates.

LET P (fig. 23) be any point whose cylindrical coordinates are r, θ, z, and let U, V, W be the displacements of the point. Then if $U+dU$, &c. be the corresponding quantities at

$$r+dr,\ \ \theta+d\theta,\ \ z+dz,$$

it follows that the actual displacements are

$$\left.\begin{aligned} &dU-Vd\theta \\ &dV+Ud\theta \\ &dW \end{aligned}\right\} \begin{aligned} &\text{along } r, \\ &\text{perpendicular to } r, \\ &\text{parallel to } z. \end{aligned}$$

Hence, if we imagine a set of auxiliary rectangular axes x', y', z' constructed at P, so as to be parallel to the directions of U, V, W, it follows that if u', v', w' be the displacements at this point referred to x', y', z', then

$$du' = dU - Vd\theta,$$

$$dv' = dV + Ud\theta,$$

$$dw' = dW,$$

also $$dx' = dr,\ \ dy' = rd\theta,\ \ dz' = dz.$$

Hence

$$\frac{du'}{dx'}dx' + \frac{du'}{dy'}dy' + \frac{du'}{dz'}dz' = \frac{dU}{dr}dr + \frac{dU}{d\theta}d\theta + \frac{dU}{dz}dz - Vd\theta;$$

therefore

$$\frac{du'}{dx'} = \frac{dU}{dr},$$

$$\frac{du'}{dy'} = \frac{dU}{rd\theta} - \frac{V}{r},$$

$$\frac{du'}{dz'} = \frac{dU}{dz}.$$

So too, in like manner,

$$\frac{dv'}{dx'} = \frac{dV}{dr},$$
$$\frac{dv'}{dy'} = \frac{dV}{r\,d\theta} + \frac{U}{r},$$
$$\frac{dv'}{dz'} = \frac{dV}{dz}.$$

And, finally,

$$\frac{dw'}{dx'} = \frac{dW}{dr}, \quad \frac{dw'}{dy'} = \frac{dW}{r\,d\theta}, \quad \frac{dw'}{dz'} = \frac{dW}{dz}.$$

Hence, the six strain components at $r\,\theta z$ are

$$\frac{dU}{dr}, \qquad \frac{dV}{r\,d\theta} + \frac{U}{r}, \qquad \frac{dW}{dz},$$
$$\frac{dV}{dz} + \frac{dW}{r\,d\theta}, \quad \frac{dU}{dz} + \frac{dW}{dr}, \quad \frac{dU}{r\,d\theta} + \frac{dV}{dr} - \frac{V}{r}.$$

Strain in polar coordinates in three dimensions.

Let us take in the first instance the same coordinates as Lamè (fig. 24)

$$r, \quad \phi = \text{latitude}, \quad \psi = \text{longitude}.$$

Then we know that if ξ, η, ζ be any vector referred to axes that move so that $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$ are the components referred to axes whose positions are derived from the former by the elementary rotations

$$\delta\theta_1, \ \delta\theta_2, \ \delta\theta_3,$$

Then the actual increments of the vector are

$$\delta\xi - \eta\delta\theta_3 + \zeta\delta\theta_2,$$
$$\delta\eta - \zeta\delta\theta_1 + \xi\delta\theta_3,$$
$$\delta\zeta - \xi\delta\theta_2 + \eta\delta\theta_1.$$

Now, in our present case,

$$\delta\theta_1 = -\delta\psi\sin\phi, \ \delta\theta_2 = -\delta\psi\cos\phi, \ \delta\theta_3 = \delta\phi.$$

Hence, the strains du', dv', dw', are now

$$\left.\begin{aligned} &dU - Vd\phi - W\cos\phi\,d\psi \\ &dV + W\sin\phi\,d\psi + Ud\phi \\ &dW + U\cos\phi\,d\psi - V\sin\phi\,d\psi \end{aligned}\right\}.$$

Hence, as now,

$$dx' = dr, \quad dy' = r d\phi, \quad dz' = r \cos\phi d\psi.$$

It follows that

$$\frac{du'}{dx'} dx' + \frac{du'}{dy'} dy' + \frac{du'}{dz'} dz'$$

$$= \frac{dU}{dr} dr + \frac{dU}{d\phi} d\phi + \frac{dU}{d\psi} d\psi - V d\phi - W \cos\phi d\psi;$$

therefore

$$\frac{du'}{dx'} = \frac{dU}{dr}, \quad \frac{du'}{dy'} = \frac{1}{r}\frac{dU}{d\phi} - \frac{V}{r}, \quad \frac{du'}{dz'} = \frac{1}{r\cos\phi}\frac{dU}{d\psi} - \frac{W}{r}.$$

Nextly,

$$\frac{dv'}{dx'} dx' + \frac{dv'}{dy'} dy' + \frac{dv'}{dz'} dz'$$

$$= \frac{dV}{dr} dr + \frac{dV}{d\phi} d\phi + \frac{dV}{d\psi} d\psi + V d\phi + W \sin\phi d\psi;$$

therefore

$$\frac{dv'}{dx'} = \frac{dV}{dr}, \quad \frac{dv'}{dy'} = \frac{1}{r}\frac{dV}{d\phi} + \frac{U}{r}, \quad \frac{dv'}{dz'} = \frac{1}{r\cos\phi}\frac{dV}{d\psi} + \frac{\tan\phi}{r} W,$$

and, lastly,

$$\frac{dw'}{dx'} dx' + \frac{dw'}{dy'} dy' + \frac{dw'}{dz'} dz'$$

$$= \frac{dW}{dr} dr + \frac{dW}{d\phi} d\phi + \frac{dW}{d\psi} d\psi + U \cos\phi d\psi - V \sin\phi d\psi;$$

therefore

$$\frac{dw'}{dx'} = \frac{dW}{dr}, \quad \frac{dw'}{dy'} = \frac{1}{r}\frac{dW}{d\phi}, \quad \frac{dw'}{dz'} = \frac{1}{r\cos\phi}\frac{dW}{d\psi} + \frac{U}{r} - \frac{V}{r}\tan\phi.$$

Hence, finally, the six component strains are

$$\frac{du'}{dx'} \equiv \frac{dU}{dr},$$

$$\frac{dv'}{dy'} \equiv \frac{U}{r} + \frac{1}{r}\frac{dV}{d\phi},$$

$$\frac{dw'}{dz'} \equiv \frac{U}{r} - \frac{\tan\phi}{r} V + \frac{1}{r\cos\phi}\frac{dW}{d\psi},$$

$$\frac{dv'}{dz'}+\frac{dw'}{dy'}=\frac{1}{r\cos\phi}\frac{dV}{d\psi}+\frac{\tan\phi}{r}W+\frac{1}{r}\frac{dW}{d\phi},$$

$$\frac{dw'}{dx'}+\frac{du'}{dz'}=\frac{1}{r\cos\phi}\frac{dU}{d\psi}+\frac{dW}{dr}-\frac{W}{r},$$

$$\frac{du'}{dy'}+\frac{dv'}{dx'}=\frac{1}{r}\frac{dV}{d\phi}-\frac{V}{r}+\frac{dV}{dr},$$

and the stresses are

$$\left(k-\frac{2n}{3}\right)\theta'+2n\frac{du'}{dx'},$$

.....

$$n\left(\frac{dv'}{dz'}+\frac{dw'}{dy'}\right),$$

.....

We now proceed to form the equations of equilbrium on both cases. We shall call these stresses

$$N_1,\ N_2,\ N_3,$$
$$T_1,\ T_2,\ T_3,$$

in both cases.

Let us consider the equilibrium of the element bounded by the edges

$$dr,\ rd\theta,\ dz \text{ (fig. 25)};$$

also let

$$\Delta_1=rd\theta\,dz,$$
$$\Delta_2=dz\,dr,$$
$$\Delta_3=dr\,rd\theta.$$

So that Δ_1, Δ_2, Δ_3 are the areas of the three faces of the element at r, θ, z.

Firstly, consider the forces on the face Δ_1, they are

$$-\Delta_1N_1,\ -\Delta_1T_3,\ -\Delta_1T_2$$

parallel respectively to x', y', z'. Those on the opposite face are parallel to the lines x', y', z' corresponding to an increment dr *in the variable* r *alone.* They are therefore parallel to x', y', z'. Hence, this pair of faces contribute stresses

$$dr\frac{d}{dr}(\Delta_1N_1) \text{ parallel to } x',$$

$$dr\frac{d}{dr}(\Delta_1T_3) \quad \text{,,} \quad y',$$

$$dr\frac{d}{dr}(\Delta_1T_2) \quad \text{,,} \quad z'.$$

Nextly, consider the face Δ_2. The stress on the side nearer the origin are $\Delta_2 T_3$, $\Delta_2 N_3$ and $\Delta_2 T_1$ parallel to $x'y'z'$.

Those on the opposite face are what these expressions become when θ is increased by $d\theta$, and their directions are parallel to the positions of $x'y'z'$ implied by the sole increment $d\theta$ of θ; in this case we then have

$$d\theta_3 = d\theta \text{ and } d\theta_1 = 0,\ d\theta_2 = 0.$$

Hence the stresses contributed by this pair of faces are

$$d\theta \frac{d}{d\theta}(\Delta_2 T_3) - d\theta\,(\Delta_2 N_2) \text{ parallel to } x',$$

$$d\theta \frac{d}{d\theta}(\Delta_2 N_2) + d\theta\,(\Delta_2 T_3) \quad \text{,,} \quad y',$$

$$d\theta \frac{d}{d\theta}(\Delta_2 T_1) \quad \text{,,} \quad z'.$$

Lastly, the stresses contributed by the faces Δ_3 are

$$dz \frac{d}{dz}(\Delta_3 T_2) \text{ parallel to } x',$$

$$dz \frac{d}{dz}(\Delta_3 T_1) \quad \text{,,} \quad y',$$

$$dz \frac{d}{dz}(\Delta_3 N_3) \quad \text{,,} \quad z'.$$

Hence, all together, the stresses give rise to

$$dr \frac{d}{dr}(\Delta_1 N_1) + d\theta \left[\frac{d}{d\theta}(\Delta_2 T_3) - \Delta_2 N_2\right] + dz \frac{d}{dz}(\Delta_3 T_2),$$

$$dr \frac{d}{dr}(\Delta_1 T_3) + d\theta \left[\frac{d}{d\theta}(\Delta_2 N_2) + \Delta_2 T_3\right] + dz \frac{d}{dz}(\Delta_3 T_1),$$

$$dr \frac{d}{dr}(\Delta_1 T_2) + d\theta \frac{d}{d\theta}(\Delta_2 T_1) + dz \frac{d}{dz}(\Delta_3 N_3).$$

If now R, Θ, Z be the external field of force per unit of mass, then the equations of motion (small) are

$$dr \frac{d}{dr}(\Delta_1 N_1) + d\theta \left[\frac{d}{d\theta}(\Delta_2 T_3) - \Delta_2 N_2\right] + dz \frac{d}{dz}(\Delta_3 T_2) + R\rho dr\, r d\theta dz = \rho dr\, r d\theta\, dz \frac{d^2 U}{dt^2},$$

$$dr \frac{d}{dr}(\Delta_1 T_3) + d\theta \left[\frac{d}{d\theta}(\Delta_2 N_2) + \Delta_2 T_3\right] + dz \frac{d}{dz}(\Delta_3 T_1) + \Theta\rho dr\, r d\theta dz = \rho dr\, r d\theta\, dz \frac{d^2 V}{dt^2},$$

$$dr\frac{d}{dr}(\Delta_1 T_2)+d\theta\left[\frac{d}{d\theta}(\Delta_2 T_1)\right] \qquad +dz\frac{d}{dz}(\Delta_3 N_3)+Z\rho dr\,rd\theta dz$$
$$=\rho\,dr\,rd\theta\,dz\frac{d^2W}{dt^2}.$$

On substituting for $\Delta_1\Delta_2\Delta_3$, and dividing out by $rdr\,d\theta\,dz$, there results

$$\frac{1}{r}\frac{d}{dr}(N_1 r)+\frac{1}{r}\frac{dT_3}{d\theta}+\frac{dT_2}{dz}-\frac{N_2}{r}+R\rho=\rho\frac{d^2U}{dt^2},$$
$$\frac{1}{r}\frac{d}{dr}(T_3 r)+\frac{1}{r}\frac{dN_2}{d\theta}+\frac{dT_1}{dz}+\frac{T_3}{r}+\Theta\rho=\rho\frac{d^2V}{dt^2},$$
$$\frac{1}{r}\frac{d}{dr}(T_2 r)+\frac{1}{r}\frac{dT_1}{d\theta}+\frac{dN_3}{dz}\qquad+Z\rho=\rho\frac{d^2W}{dt^2},$$

or
$$\left.\begin{aligned}
&\frac{dN_1}{dr}+\frac{1}{r}\frac{dT_3}{d\theta}+\frac{dT_2}{dz}+\frac{N_1-N_2}{r}+R\rho=\rho\frac{d^2U}{dt^2}\\
&\frac{dT_3}{dr}+\frac{1}{r}\frac{dN_2}{d\theta}+\frac{dT_1}{dz}+\frac{2T_3}{r}+\Theta\rho=\rho\frac{d^2V}{dt^2}\\
&\frac{dT_2}{dr}+\frac{1}{r}\frac{dT_1}{d\theta}+\frac{dN_3}{dz}+\frac{T_2}{r}+Z\rho=\rho\frac{d^2W}{dt^2}
\end{aligned}\right\}.$$

In polar coordinates in three dimensions we have to proceed in a like manner, here

$$\Delta_1=r\cos\phi\,d\phi\,rd\psi,$$
$$\Delta_2=r\cos\phi\,d\psi\,dr,$$
$$\Delta_3=rd\phi\,dr.$$

Now, consider the stresses contributed by the pair of faces Δ_1 (fig. 26).

Here $d\theta_1=0$, $d\theta_2=0$, $d\theta_3=0$, and, therefore, the components of this contribution are

$$dr\frac{d}{dr}(N_1\Delta_1),\quad dr\frac{d}{dr}(T_3\Delta_1),\quad dr\frac{d}{dr}(T_2\Delta_1).$$

Secondly, consider the component contributions of the pair of faces Δ_2; in this case $d\theta_1=0$, $d\theta_2=0$, $d\theta_3=d\phi$, hence the components wanted are

$$d\phi\frac{d}{d\phi}(T_3\Delta_2)-d\theta_3(N_2\Delta_2)+d\theta_2(T_1\Delta_2),$$
$$d\phi\frac{d}{d\phi}(N_2\Delta_2)-d\theta_1(T_1\Delta_2)+d\theta_3(T_3\Delta_2),$$
$$d\phi\frac{d}{d\phi}(T_1\Delta_2)-d\theta_2(T_3\Delta_2)+d\theta_1(N_2\Delta_2),$$

that is

$$\left.\begin{aligned} &d\phi \frac{d}{d\phi}(T_3\Delta_2) - d\phi\,(N_2\Delta_2) \\ &d\phi \frac{d}{d\phi}(N_2\Delta_2) + d\phi\,(T_3\Delta_2) \\ &d\phi \frac{d}{d\phi}(T_1\Delta_2) \end{aligned}\right\}.$$

Lastly, consider the pair of faces Δ_3... the stresses on the nearer face are

$$-T_2\Delta_3,\ -T_1\Delta_2,\ -N_3\Delta_3,$$

and here

$$\begin{aligned} d\theta_1 &= -d\psi \sin\phi, \\ d\theta_2 &= -d\psi \cos\phi, \\ d\theta_3 &= \ 0. \end{aligned}$$

Hence, the corresponding components are

$$d\psi \frac{d}{d\psi}(T_2\Delta_3) - d\psi \cos\phi\,(N_3\Delta_3),$$

$$d\psi \frac{d}{d\psi}(T_1\Delta_3) + d\psi \sin\phi\,(N_3\Delta_3),$$

$$d\psi \frac{d}{d\psi}(N_3\Delta_3) + d\psi \cos\phi\,(T_2\Delta_3) - d\psi \sin\phi\,(T_1\Delta_3).$$

Hence, adding all three results together, the x', y', z' resultant components are

$$dr \frac{d}{dr}(N_1\Delta_1) + d\phi \left[\frac{d}{d\phi}(T_3\Delta_2) - N_2\Delta_2\right] + d\psi \left[\frac{d}{d\psi}(T_2\Delta_3) - \cos\phi\, N_3\Delta_3\right],$$

$$dr \frac{d}{dr}(T_3\Delta_1) + d\phi \left[\frac{d}{d\phi}(N_2\Delta_2) + T_3\Delta_2\right] + d\psi \left[\frac{d}{d\psi}(T_1\Delta_3) + \sin\phi\, N_3\Delta_3\right],$$

$$dr \frac{d}{dr}(T_2\Delta_1) + d\phi \left[\frac{d}{d\phi}\, T_1\Delta_2\right] + d\psi \left[\frac{d}{d\psi}(N_3\Delta_3) + \cos\phi\, T_2\Delta_3 - \sin\phi\, T_1\Delta_3\right],$$

Now substituting for

$$\Delta_1 \equiv r^2 \cos\phi \, d\phi \, d\psi,$$
$$\Delta_2 \equiv r \cos\phi \, dr \, d\psi,$$
$$\Delta_3 \equiv r \, d\phi \, dr.$$

The above become, when divided out by $r^2 \cos\phi \, d\phi \, d\psi \, dr$,

$$\frac{1}{r^2}\frac{d}{dr}(N_1 r^2) + \frac{1}{r\cos\phi}\frac{d}{d\phi}(T_3 \cos\phi) + \frac{1}{r\cos\phi}\frac{dT_3}{d\psi} - \frac{N_2 + N_3}{r},$$

$$\frac{1}{r^2}\frac{d}{dr}(T_3 r^2) + \frac{1}{r\cos\phi}\frac{d}{d\phi}(N_2 \cos\phi) + \frac{1}{r\cos\phi}\frac{dT_1}{d\psi} + \frac{T_3 + N_3 \tan\phi}{r},$$

$$\frac{1}{r^2}\frac{d}{dr}(T_2 r^2) + \frac{1}{r\cos\phi}\frac{d}{d\phi}(T_1 \cos\phi) + \frac{1}{r\cos\phi}\frac{dN_3}{d\psi} + \frac{T_2 - T_1 \tan\phi}{r}.$$

Hence the final equations of motion are

$$\frac{dN_1}{dr} + \frac{1}{r}\frac{dT_3}{d\phi} + \frac{1}{r\cos\phi}\frac{dT_2}{d\psi} + \frac{2N_1 - N_2 - N_3 - T_3\tan\phi}{r} + R\rho = \rho\frac{d^2U}{dt^2},$$

$$\frac{dT_3}{dr} + \frac{1}{r}\frac{dN_2}{d\phi} + \frac{1}{r\cos\phi}\frac{dT_1}{d\psi} + \frac{3T_3 + (N_3 - N_2)\tan\phi}{r} + \Phi\rho = \rho\frac{d^2V}{dt^2},$$

$$\frac{dT_2}{dr} + \frac{1}{r}\frac{dT_1}{d\phi} + \frac{1}{r\cos\phi}\frac{dN_3}{d\psi} + \frac{3T_2 - 2T_1\tan\phi}{r} + \Psi\rho = \rho\frac{d^2W}{dt^2}.$$

A few special cases are worth recalling.

(i) *Cylindrical radial strain only.*

Here the formulæ reduce to very simple shapes, in fact, the principal strains at any point are

$$\frac{dU}{dr}, \ \frac{U}{r}, \ 0, \ 0, \ 0, \ 0;$$

and, therefore,
$$\theta = \frac{dU}{dr} + \frac{U}{r},$$

where θ is the cubic dilatation.

The stresses are likewise given by

$$N_1 = (k - \tfrac{2}{3}n)\,\theta + 2n\frac{dU}{dr},$$

$$N_2 = (k - \tfrac{2}{3}n)\,\theta + 2n\frac{U}{r},$$

$$N_3 = (k - \tfrac{2}{3}n)\,\theta,$$

$$T_1 = 0, \ T_2 = 0, \ T_3 = 0,$$

and the equation of radial vibrations under radially applied force is

$$\frac{dN_1}{dr} + \frac{N_1 - N_2}{r} + \rho R = \rho \frac{d^2U}{dt^2},$$

so that on substituting for the stresses their expressions in terms of the strains there results

$$(k + \tfrac{4}{3}n)\left(\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - \frac{U}{r^2}\right) + \rho R = \rho \frac{d^2U}{dt^2};$$

or, what is the same thing,

$$(k + \tfrac{4}{3}n)\frac{d\theta}{dr} + \rho R = \rho \frac{d^2U}{dt^2}.$$

(ii) *Cylindrical transversal strain only.*

Here we mean that each particle is merely displaced along the circle on which it lies, and that there is no variation of strain on passing along the axis. In this case we have the strains given by

$$0, 0, 0, 0, 0, \frac{dV}{dr} - \frac{V}{r},$$

so that $\theta = 0$, and the equation of stress would be

$$N_1 = 0,\ N_2 = 0,\ N_3 = 0,\ T_1 = 0,\ T_2 = 0,\ T_3 = n\left(\frac{dV}{dr} - \frac{V}{r}\right);$$

also the equation of vibrations under forces applied perpendicularly to the meridian plane is

$$\frac{dT_3}{dr} + \frac{2T_3}{r} + \rho S = \rho \frac{d^2V}{dt^2};$$

on substituting for the stress its strain equivalent we have

$$n\left(\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2}\right) + \rho S = \rho \frac{d^2V}{dt^2}.$$

It is perhaps worth noticing that the equation in V in this case is identical with that in U in the preceding case. The equation of equilibrium in this case is interesting. (See Math. Tripos, 1882, Jan. 17, 9–12).

(iii). *Torsion uniformly distributed in a plane perpendicular to the axis.*

Here we have

$$U = 0, \quad V = r\tau, \quad W = 0,$$

where τ is the torsion at the section whose distance along the

axis is z and of course a function of z only. The strains here are

$$0,\ 0,\ 0,\ r\frac{d\tau}{dz},\ 0,\ 0,$$

and the stresses all zero except T_1 which $= nr\frac{d\tau}{dz}$, and the equation of vibration is merely

$$\frac{dT_1}{dz} = \rho\frac{d^2V}{dt^2}$$

under no forces, or

$$nr\frac{d^2\tau}{dz^2} = \rho r\frac{d^2\tau}{dt^2},$$

or

$$n\frac{d^2\tau}{dz^2} = \rho\frac{d^2\tau}{dt^2}.$$

(iv) *Spherical radial strain only.*

This case is the analogue in three dimensions of case (i), and the principal strains are now

$$\frac{dU}{dr},\ \frac{U}{r},\ \frac{U}{r},\ 0,\ 0,\ 0,$$

so that

$$\theta = \frac{dU}{dr} + 2\frac{U}{r},$$

so that the stresses become

$$N_1 = (k - \tfrac{2}{3}n)\,\theta + 2n\frac{dU}{dr},$$

$$N_2 = (k - \tfrac{2}{3}n)\,\theta + 2n\frac{U}{r} = N_3,$$

and

$$T_1 = T_2 = T_3 = 0;$$

further, the equation of radial vibrations is

$$\frac{dN_1}{dr} + \frac{2N_1 - N_2 - N_3}{r} + \rho R = \rho\frac{d^2U}{dt^2};$$

on substitution we arrive at

$$(k + \tfrac{4}{3}n)\left(\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} - \frac{2U}{r^2}\right) + \rho R = \rho\frac{d^2U}{dt^2};$$

or, what is the same thing,

$$(k + \tfrac{4}{3}n)\frac{d\theta}{dr} + \rho R = \rho\frac{d^2U}{dt^2}.$$

ON THE EQUILIBRIUM OF A BENT PLATE.

By *R. R. Webb, M.A.*

In what follows we shall, for simplicity, confine our attention to the cases having

(α) isotropic elastic properties,

(β) infinitely small deviations from an unstrained plate of uniform thickness.

When these conditions hold good, and if there be no forces in the plane of the plate other than those consequent on strain, it follows that sheets of particles, whose planes are parallel to the plane faces of the unstrained plate become, when strained, surfaces commonly called parallel surfaces, or, at any rate, the succession of elements of these sheets cut out by the normals along the boundary of a closed curve on either face, may be considered to be *parallel* surfaces.

In the first place, consider a small element of the strained plate, and let it be bounded by the normals that meet the surfaces of the elementary sheets in two pairs of elements of lines of curvature, and suppose the strain everywhere small.

Now consider the shape of this element (fig. 27), all the normals at points on the surfaces of these elementary plates pass through two focal lines.

Further, as the strain is small, we can use the principle of superposition, and we therefore have to discuss the effects of two coexistent cylindrical strains, and we merely add the effects.

Now if we consider the plate as made of filaments themselves elements of the lines of curvature of the particular sheets on which they respectively lie, it follows that the whole state of stress must be ultimately that due to the tensions of two sets of fibres, and as each of these elementary tensions is perpendicular to one or other of the normal faces of the element, we conclude just as in the treatment of a rod, that the whole state of stress in this element is simply that due to couples along axes which are *tangents* to the lines of curvature of the central sheet; also if the size and shape of this element of this central sheet be such that the lengths of the lines of curvature are ds_1, ds_2, and the radii of curvature of the normal sections ρ_1, ρ_2, then the couples are $\frac{EI}{\rho_1} ds_2$ and $\frac{EI}{\rho_2} ds_1$ about

ds_2 and ds_1 respectively, I being the moment of inertia of a normal section of one unit of length about a central line parallel to its length.

In order to see this result, just consider the normal section $ABCD$ (fig. 28) having AB and CD in the two opposite faces of the plate, and each parallel to ds_2.

Then the stresses normal to this section are of magnitude

$$E\frac{\eta}{\rho_1} \text{ unit of area,}$$

η denoting the distance of the fibre from ds_2, the element of the line of curvature in which $ABCD$ meets the central surface. These elementary tensions give, as usual, a couple

$$\iint \delta A E \frac{\eta^2}{\rho_1} \text{ about } ds_2;$$

i.e., $\frac{EI}{\rho_1} ds_2$, where I is the moment of inertia of $ABCD$ about EF, on the supposition that EF is unit length.

Hence, we may view this principal state of stress as being determined by the principal components

$$\left.\begin{aligned} &\frac{EI}{\rho_2} \text{ unit length about } ds_1, \\ \text{and} \quad &\frac{EI}{\rho_1} \text{ unit length about } ds_2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\text{I.}$$

We shall nextly show that if any other than principal normal sections be taken there will be normal couples; *i.e.*, couples on $ABCD$ having ds_1, as well as ds_2 for the direction of their axis.

For this purpose, consider a small element of a synclastic surface referred to the normal and directions of the principal curvatures as axes. Suppose we fix our attention on the portion ABC (fig. 29), where AOC makes an angle ϕ with Ox. Simple inspection shows that as the surface is synclastic and OZ is the inward drawn normal, the stress couples on this element are

$\frac{EI}{\rho_2}$ per unit length of BC with axis parallel to Ox,

and $\frac{EI}{\rho_1}$ per unit length of AB with axis parallel to Oy.

Let now the stress couples on the element across the section AOC be (i) K per unit of length with axis along AOC, (ii) P per unit of length with axis perpendicular to AOC, and

such that a screw from the direction of K to that of P is in the same sense as one from Ox to Oy.

Then, by considering the element ABC, we have the following equations of couples

$$\left.\begin{aligned} K + EI\left(\frac{\cos^2\phi}{\rho_2} - \frac{\sin^2\phi}{\rho_1}\right) &= 0 \\ P + EI\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin\phi\cos\phi &= 0 \end{aligned}\right\} \begin{aligned} &\text{.....(1),} \\ &\text{.....(2);} \end{aligned}$$

this gives the action at the section ϕ on the portion that lies beyond it; it easily appears that (1) gives the correct results in the cases of $\phi = 0$, $\phi = \frac{1}{2}\pi$, defining the principal sections.

Next, take two new axes Ox', Oy' (fig. 30) obtained by turning Ox, Oy through an angle ϕ; then the actions across the sections Ox', Oy' on an element that succeeds them are for Ox'

$$-EI\left(\frac{\cos^2\phi}{\rho_2} + \frac{\sin^2\phi}{\rho_1}\right) \text{ with axis parallel to } Ox',$$

$$-EI\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin\phi\cos\phi \text{ with axis parallel to } Oy'.$$

So, likewise, for the action across Oy' on the succeeding element, we should merely have to put $\frac{1}{2}\pi + \phi$ for ϕ; this gives

$$-EI\left(\frac{\sin^2\phi}{\rho_2} + \frac{\cos^2\phi}{\rho_1}\right) \text{ with axis parallel to } Oy',$$

$$+EI\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin\phi\cos\phi \text{ with axis parallel to } -Ox',$$

$$\text{or } -EI\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin\phi\cos\phi \text{ with axis parallel to } Ox'.$$

Hence, to obtain the action along Oy' on the element contained between Ox' and Oy', we should of course have to change the signs of these expressions, and get, finally, for the action on the element between the axes $Ox'.Oy'$,

$$\left.\begin{aligned} &-EI\left(\frac{\cos^2\phi}{\rho_2} + \frac{\sin^2\phi}{\rho_1}\right) \text{ parallel to } Ox' \\ &-EI\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin\phi\cos\phi \text{ parallel to } Oy' \end{aligned}\right\} \text{across } Ox',$$

$$\left.\begin{aligned} &+ EI\left(\frac{\sin^2\phi}{\rho_2} + \frac{\cos^2\phi}{\rho_1}\right) \text{ parallel to } Oy' \\ &+ EI\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\sin\phi\cos\phi \text{ parallel to } Ox' \end{aligned}\right\} \text{across } Oy'.$$

We now proceed to express these differently.

Let z be the displacement of any point from the plane xOy; then $\frac{dz}{dx}$, $\frac{dz}{dy}$ are in the limit such that their squares can be neglected, and in the end solid geometry tells us that

$$\frac{1}{\rho_1} = \frac{d^2z}{dx^2} \text{ and } \frac{1}{\rho_2} = \frac{d^2z}{dy^2},$$

while here

$$\frac{d^2z}{dx\,dy} = 0.$$

Again

$$\left.\begin{aligned} \frac{d}{dx'} &= \cos\phi\,\frac{d}{dx} + \sin\phi\,\frac{d}{dy} \\ \frac{d}{dy'} &= -\sin\phi\,\frac{d}{dx} + \cos\phi\,\frac{d}{dy} \end{aligned}\right\};$$

therefore

$$\frac{d^2z}{dx'^2} = \cos^2\phi\,\frac{d^2z}{dx^2} + \sin^2\phi\,\frac{d^2z}{dy^2},$$

$$\frac{d^2z}{dy'^2} = \sin^2\phi\,\frac{d^2z}{dx^2} + \cos^2\phi\,\frac{d^2z}{dy^2},$$

$$\frac{d^2z}{dx'\,dy'} = \sin\phi\cos\phi\left(\frac{d^2z}{dy^2} - \frac{d^2z}{dx^2}\right).$$

Hence, substituting, we get that the couple stresses across Ox' on the succeeding element are

$$\left.\begin{aligned} &- EI\,\frac{d^2z}{dy'^2} \text{ with axis parallel to } Ox' \\ &+ EI\,\frac{d^2z}{dx'dy'} \text{ with axis parallel to } Oy' \end{aligned}\right\},$$

and across Oy' on the succeeding element

$$+ EI\,\frac{d^2z}{dx'dy'} \text{ with axis parallel to } Ox'$$

$$- EI\,\frac{d^2z}{dx'^2} \text{ with axis parallel to } Oy'$$

Hence, considering the element between the axes Ox' and Oy', the action at all points of Oy' must be

$$-EI\frac{d^2z}{dx'dy'} \text{ with axis parallel to } Ox',$$

$$+EI\frac{d^2z}{dx'^2} \text{ with axis parallel to } Oy'.$$

We may now drop the dashes, calling now

$$\left.\begin{aligned} M &\equiv EI\frac{d^2z}{dx^2} \\ N &\equiv EI\frac{d^2z}{dxdy} \\ L &\equiv EI\frac{d^2z}{dy^2} \end{aligned}\right\}.$$

Then fig. 31 gives the state of stress on the element per unit length of the respective edges, when the single dash indicates that x becomes $x+dx$, and the double dash that y has become $y+dy$.

Suppose now that the only externally impressed force is Z per unit area perpendicular to the plate, while P and Q are the component unpressed couples about axes in the plate. Further, let α, β be the shears parallel to Oz across the normal sections parallel to $Ox\,Oy$. Then, just as in Thomson and Tait, *Nat. Phil.*, §§ 644, we get

$$\left.\begin{aligned} Z+\frac{d\alpha}{dx}+\frac{d\beta}{dy} &= 0 \\ P+\beta+\frac{dL}{dy}+\frac{dN}{dx} &= 0 \\ Q-\alpha-\frac{dN}{dy}-\frac{dM}{dx} &= 0 \end{aligned}\right\};$$

and hence it follows that

$$\frac{d^2L}{dy^2}+2\frac{d^2N}{dxdy}+\frac{d^2M}{dx^2}+\frac{dP}{dy}-\frac{dQ}{dx}-Z=0,$$

or
$$EI\left(\frac{d^4z}{dx^4}+2\frac{d^2z}{dx^2dy^2}+\frac{d^4z}{dy^4}\right)=Z-\frac{dP}{dy}+\frac{dQ}{dx},$$

which determines the deflexion.

ON A DETERMINANT FORMED BY BORDERING THE PRODUCT OF TWO DETERMINANTS.

By *Thomas Muir, M.A., F.R.S.E.*

1. If we take two determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \text{ or } D, \quad \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \text{ or } \Delta,$$

and form their product

$$\begin{vmatrix} a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 & a_1\beta_1 + a_2\beta_2 + a_3\beta_3 & a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 \\ b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 & b_1\beta_1 + b_2\beta_2 + b_3\beta_3 & b_1\gamma_1 + b_2\gamma_2 + b_3\gamma_3 \\ c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 & c_1\beta_1 + c_2\beta_2 + c_3\beta_3 & c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3 \end{vmatrix}$$

denoting it for shortness sake by

$$\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix},$$

then the determinant to be considered is

$$\begin{vmatrix} 0 & \alpha_0 & \beta_0 & \gamma_0 \\ a_0 & A_1 & A_2 & A_3 \\ b_0 & B_1 & B_2 & B_3 \\ c_0 & C_1 & C_2 & C_3 \end{vmatrix}.$$

2. It is clearly equal to

$$\begin{vmatrix} -1 & \alpha_0 & \beta_0 & \gamma_0 \\ a_0 & A_1 & A_2 & A_3 \\ b_0 & B_1 & B_2 & B_3 \\ c_0 & C_1 & C_2 & C_3 \end{vmatrix} + A_1B_2C_3|.$$

Transforming the first of these determinants by increasing each element of the second row by a_0 times the corresponding element of the first row, each element of the third row by b_0 times the corresponding element of the first row, and so on, we obtain

$$-\begin{vmatrix} a_0\alpha_0 + A_1 & a_0\beta_0 + A_2 & a_0\gamma_0 + A_3 \\ b_0\alpha_0 + B_1 & b_0\beta_0 + B_2 & b_0\gamma_0 + B_3 \\ c_0\alpha_0 + C_1 & c_0\beta_0 + C_2 & c_0\gamma_0 + C_3 \end{vmatrix} + |A_1B_2C_3|,$$

which we know is equal to

$$-\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} + |A_1B_2C_3|,$$

and therefore equal to

$$-\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix} \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & \beta_1 & \beta_2 \\ \gamma_0 & \gamma_1 & \gamma_2 \end{vmatrix} - \begin{vmatrix} a_0 & a_1 & a_3 \\ b_0 & b_1 & b_3 \\ c_0 & c_1 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_0 & \alpha_1 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_3 \end{vmatrix}$$
$$- \begin{vmatrix} a_0 & a_2 & a_3 \\ b_0 & b_2 & b_3 \\ c_0 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_0 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_2 & \gamma_3 \end{vmatrix}.$$

We thus have a result which may be stated as follows:

$$-\begin{vmatrix} 0 & \alpha_0 & \beta_0 & \gamma_0 \\ a_0 & & & \\ b_0 & \times & |a_1b_2c_3| & \\ c_0 & & |\alpha_1\beta_2\gamma_3| & \end{vmatrix}$$
$$= |a_0b_1c_2||\alpha_0\beta_1\gamma_2| + |a_0b_1c_3||\alpha_0\beta_1\gamma_3| + |a_0b_2c_3||\alpha_0\beta_2\gamma_3| \ldots (\mathrm{I}).$$

The corresponding result for determinants of any higher order is evident.

3. In the second place, there is an elegant symbolical expression for either of the members of (I) which is worthy of attention. Expanding the left-hand member according to binary products of the elements of the first row and first column, we have

$$a_0\alpha_0 \begin{vmatrix} b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix} \begin{vmatrix} \beta_0 & \beta_1 & \beta_2 \\ \gamma_0 & \gamma_1 & \gamma_2 \end{vmatrix} - a_0\beta_0 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} + \ldots$$

which is seen to be symbolically resolvable into the two factors

$$a_0 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - b_0 \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + c_0 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

$$\alpha_0 \begin{vmatrix} \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} - \beta_0 \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} + \gamma_0 \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{vmatrix}.$$

Hence, denoting the matrix resulting from the deletion of the r^{th} row of D by $D^{(r}$ we have the theorem

$$-\begin{vmatrix} 0 & \alpha_0 & \beta_0 & \gamma_0 \\ a_0 & & & \\ b_0 & & D \times \Delta & \\ c_0 & & & \end{vmatrix}$$

$$= \{a_0 D^{(1} - b_0 D^{(2} + c_0 D^{(3}\} \{\alpha_0 \Delta^{(1} - \beta_0 \Delta^{(2} + \gamma_0 \Delta^{(3}\} \ldots \text{(II)}.$$

4. Numerous special cases of these identities need not be referred to. Only one is markedly of interest. This is the case where D and Δ are the same zero-axial skew determinant, and $a_0 = \alpha_0$, $b_0 = \beta_0$, $c_0 = \gamma_0$.

Taking even-ordered determinants first, let

$$D = \Delta = \begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix};$$

then from (I) we have

$$-\begin{vmatrix} 0 & x_1 & x_2 & x_3 & x_4 \\ x_1 & a^2+b^2+c^2 & bd+ce & -ad+cf & -ae-bf \\ x_2 & bd+ce & a^2+d^2+e^2 & ab+ef & ac-df \\ x_3 & -ad+cf & ab+ef & b^2+d^2+f^2 & bc+de \\ x_4 & -ae-bf & ac-df & bc+de & c^2+e^2+f^2 \end{vmatrix} \text{ or D, say,}$$

$$= \begin{vmatrix} x_1 & a & b & c \\ x_2 & 0 & d & e \\ x_3 & -d & 0 & f \\ x_4 & -e & -f & 0 \end{vmatrix}^2 + \begin{vmatrix} x_1 & 0 & b & c \\ x_2 & -a & d & e \\ x_3 & -b & 0 & f \\ x_4 & -c & -f & 0 \end{vmatrix}^2$$

$$+ \begin{vmatrix} x_1 & 0 & a & c \\ x_2 & -a & 0 & e \\ x_3 & -b & -d & f \\ x_4 & -c & -e & 0 \end{vmatrix}^2 + \begin{vmatrix} x_1 & 0 & a & b \\ x_2 & -a & 0 & d \\ x_3 & -b & -d & 0 \\ x_4 & -c & -e & -f \end{vmatrix}^2.$$

But each of the last four determinants is, or is transformable into, a bordered zero-axial skew determinant, and therefore by Cayley's theorem is expressible as the product of two pfaffians.

We thus obtain

$$-D = \left|\begin{matrix} a & b & c \\ & d & e \\ & & f \end{matrix}\right|^2 \left|\begin{matrix} x_2 & x_3 & x_4 \\ & d & e \\ & & f \end{matrix}\right|^2 + \left|\begin{matrix} a & b & c \\ & d & e \\ & & f \end{matrix}\right|^2 \left|\begin{matrix} x_1 & x_3 & x_4 \\ & b & c \\ & & f \end{matrix}\right|^2$$

$$+ \left|\begin{matrix} a & b & c \\ & d & e \\ & & f \end{matrix}\right|^2 \left|\begin{matrix} x_1 & x_2 & x_4 \\ & a & c \\ & & e \end{matrix}\right|^2 + \left|\begin{matrix} a & b & c \\ & d & e \\ & & f \end{matrix}\right|^2 \left|\begin{matrix} a & b & x_1 \\ & d & x_2 \\ & & x_3 \end{matrix}\right|^2;$$

and therefore

$$-D = \left|\begin{matrix} a & b & c \\ & d & e \\ & & f \end{matrix}\right|^2$$

$$\times \left\{ \left|\begin{matrix} x_2 & x_3 & x_4 \\ & d & e \\ & & f \end{matrix}\right|^2 + \left|\begin{matrix} x_1 & x_3 & x_4 \\ & b & c \\ & & f \end{matrix}\right|^2 + \left|\begin{matrix} x_1 & x_2 & x_4 \\ & a & c \\ & & e \end{matrix}\right|^2 + \left|\begin{matrix} x_1 & x_2 & x_3 \\ & a & b \\ & & d \end{matrix}\right|^2 \right\}$$

.....(III).

It is thus seen that the original determinant D or Δ is a factor of D.

The law of formation of the set of four pfaffians in (III) may be stated as follows: The first frame-line* of the r^{th} pfaffian is got by deleting the r^{th} element of the row

$$x_1 \quad x_2 \quad x_3 \quad x_4,$$

and the remaining elements are got by deleting the r^{th} frame-line of the pfaffian

$$\left|\begin{matrix} a & b & c \\ & d & e \\ & & f \end{matrix}\right|.$$

When the original determinant is of odd order the result is different in form though perfectly similarly obtained. For example, taking the *fifth* order, the theorem is

If $$D = \left|\begin{matrix} 0 & a & b & c & d \\ -a & 0 & e & f & g \\ -b & -e & 0 & h & k \\ -c & -f & -h & 0 & l \\ -d & -g & -k & -l & 0 \end{matrix}\right|$$

* The *frame-lines* of the first pfaffian in (III) for example, consist of the elements

$$a, b, c\,;\ a, d, e\,;\ b, d, f\,;\ c, e, f.$$

Some such word, to correspond with *row* and *column* in determinants, is absolutely essential.

then

$$-\begin{vmatrix} 0 & x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1 & & & & & \\ x_2 & & & & & \\ x_3 & & & D^2 & & \\ x_4 & & & & & \\ x_5 & & & & & \end{vmatrix} = \begin{vmatrix} a & b & c & d & x_1 \\ & e & f & g & x_2 \\ & & h & k & x_3 \\ & & & l & x_4 \\ & & & & x_5 \end{vmatrix}^2$$

$$\times \left\{ \begin{vmatrix} e & f & g \\ & h & k \\ & & l \end{vmatrix}^2 + \begin{vmatrix} b & c & d \\ & h & k \\ & & l \end{vmatrix}^2 + \begin{vmatrix} a & c & d \\ & f & g \\ & & l \end{vmatrix}^2 + \begin{vmatrix} a & b & d \\ & e & g \\ & & k \end{vmatrix}^2 + \begin{vmatrix} a & b & c \\ & e & f \\ & & h \end{vmatrix}^2 \right\}$$

.....(IV)

where the five last-written pfaffians are got by deleting in succession the five frame-lines of the quasi-pfaffian

$$\begin{vmatrix} a & b & c & d \\ & e & f & g \\ & & h & k \\ & & & l \end{vmatrix}.$$

Beechcroft, Bishopton, N.B.
27*th Feb.*, 1882.

ON SOME DETERMINANTS WHOSE ELEMENTS ARE RATIONAL FRACTIONS.

By *R. F. Scott, M.A.*

1. Let

$$f(x) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$
$$= x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots \pm p_n,$$

$$\frac{f(x)}{x-\alpha_k} = x^{n-1} - p_{k1} x^{n-2} + p_{k2} x^{n-3} - \ldots \mp p_{kn-1},$$

and let $$P = f(x_1) f(x_2) \ldots f(x_n).$$

Then if

$$\Delta = \begin{vmatrix} \frac{1}{x_1-\alpha_1}, & \frac{1}{x_1-\alpha_2}, & \cdots \frac{1}{x_1-\alpha_n} \\ \frac{1}{x_2-\alpha_1}, & \frac{1}{x_2-\alpha_2}, & \cdots \frac{1}{x_2-\alpha_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{x_n-\alpha_1}, & \frac{1}{x_n-\alpha_2}, & \cdots \frac{1}{x_n-\alpha_n} \end{vmatrix},$$

we have

$$\Delta . P = \begin{vmatrix} \frac{f(x_1)}{x_1-\alpha_1}, & \frac{f(x_1)}{x_1-\alpha_2}, & \cdots \frac{f(x_1)}{x_1-\alpha_n} \\ \frac{f(x_2)}{x_2-\alpha_1}, & \frac{f(x_2)}{x_2-\alpha_2}, & \cdots \frac{f(x_2)}{x_2-\alpha_n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{f(x_n)}{x_n-\alpha_1}, & \frac{f(x_n)}{x_n-\alpha_2}, & \cdots \frac{f(x_n)}{x_n-\alpha_n} \end{vmatrix}.$$

The determinant on the right is clearly the product of the two determinants

$$\begin{vmatrix} x_1^{n-1}, & x_1^{n-2}, & \ldots x_1, & 1 \\ x_2^{n-1}, & x_2^{n-2}, & \ldots x_2, & 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n^{n-1}, & x_n^{n-2}, & \ldots x_n, & 1 \end{vmatrix},$$

$$\begin{vmatrix} 1, & -p_{11}, & p_{12}, & \ldots \mp p_{1n-1} \\ 1, & -p_{21}, & p_{22}, & \ldots \mp p_{2n-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 1, & -p_{n1}, & p_{n2}, & \ldots \mp p_{nn-1} \end{vmatrix}.$$

The first of these determinants is $\zeta^{\frac{1}{2}}(x_1, x_2, \ldots x_n)$.
The second determinant is clearly equal to

$$\begin{vmatrix} p_{1n-1}, \ldots p_{12}, & p_{11}, & 1 \\ p_{2n-1}, \ldots p_{22}, & p_{21}, & 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ p_{nn-1}, \ldots p_{n2}, & p_{n1}, & 1 \end{vmatrix}.$$

Since each column in the determinant is multiplied by the requisite power of -1 to bring it into its new position.

A little consideration shews that this last determinant is the same as

$$\frac{d(p_n, p_{n-1}, \dots p_2, p_1)}{d(\alpha_1, \alpha_2, \dots \alpha_{n-1}, \alpha_n)}$$

$$= (-1)^{\frac{1}{2}\{n(n-1)\}} \frac{d(p_n, p_{n-1}, \dots p_1)}{d(\alpha_n, \alpha_{n-1}, \dots \alpha_1)}$$

$$= (-1)^{\frac{1}{2}\{n(n-1)\}} \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \dots \alpha_n).$$

Hence

$$\Delta . P = (-1)^{\frac{1}{2}\{n(n-1)\}} \zeta^{\frac{1}{2}}(x_1, x_2, \dots x_n) \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \dots \alpha_n).$$

This theorem is due to Cauchy (*Exerc. d'anal.*, ii. p. 154). His method of proof (which is given in Baltzer's *Treatise on Determinants*, ed. 5, p. 103) consisted in verifying the final result.

2. The same method applies to several determinants of a similar kind. Take, for example, that considered by Joachimsthal (*De aequationibus quarti et sexti gradus* &c., Crelle, liii. p. 169),

$$D = \begin{vmatrix} \frac{1}{x_1 - \alpha_1}, & \frac{1}{x_1 - \alpha_2}, \dots & \frac{1}{x_1 - \alpha_n}, & 1 \\ \frac{1}{x_2 - \alpha_1}, & \frac{1}{x_2 - \alpha_2}, \dots & \frac{1}{x_2 - \alpha_n}, & 1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{1}{x_{n+1} - \alpha_1}, & \frac{1}{x_{n+1} - \alpha_2}, \dots & \frac{1}{x_{n+1} - \alpha_n}, & 1 \end{vmatrix},$$

Then if $P = f(x_1) f(x_2) \dots f(x_{n+1})$,

$$D . P = \begin{vmatrix} \frac{f(x_1)}{x_1 - \alpha_1}, & \frac{f(x_1)}{x_1 - \alpha_2}, \dots & \frac{f(x_1)}{x_1 - \alpha_n}, & f(x_1) \\ \frac{f(x_2)}{x_2 - \alpha_1}, & \frac{f(x_2)}{x_2 - \alpha_2}, \dots & \frac{f(x_2)}{x_2 - \alpha_n}, & f(x_2) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{f(x_{n+1})}{x_{n+1} - \alpha_1}, & \frac{f(x_{n+1})}{x_{n+1} - \alpha_2}, \dots & \frac{f(x_{n+1})}{x_{n+1} - \alpha_n}, & f(x_{n+1}) \end{vmatrix}.$$

This last determinant is the product of the two determinants

$$\begin{vmatrix} x_1^n, & x_1^{n-1}, \dots & x_1, & 1 \\ x_2^n, & x_2^{n-1}, \dots & x_2, & 1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_{n+1}^n, & x_{n+1}^{n-1}, \dots & x_{n+1}, & 1 \end{vmatrix}.$$

and $$\begin{vmatrix} 0, & 1, & -p_{11}, & p_{12}, & \dots \mp p_{1n-1} \\ 0, & 1, & -p_{21}, & p_{22}, & \dots \mp p_{2n-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 1, & -p_1, & p_2, & -p_3, & \dots \pm p_n \end{vmatrix},$$

the first of which is equal to

$$\zeta^{\frac{1}{2}}(x_1, x_2, \dots x_{n+1}),$$

and the second equal to

$$(-1)^{\frac{1}{2}\{n(n+1)\}}\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \dots \alpha_n).$$

Hence

$$D.P = (-1)^{\frac{1}{2}\{n(n+1)\}}\zeta^{\frac{1}{2}}(x_1, x_2, \dots x_{n+1})\,\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \dots \alpha_n).$$

3. If we compound the two determinants

$$\begin{vmatrix} x_1^r + x_1^n, & x_1^{n-1}, & \dots & x_1, & 1 \\ x_2^r + x_2^n, & x_2^{n-1}, & \dots & x_2, & 1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_{n+1}^r + x_{n+1}^n, & x_{n+1}^{n-1}, & \dots & x_{n+1}, & 1 \end{vmatrix},$$

$$\begin{vmatrix} 0, & 1, & -p_{11}, & p_{12}, & \dots \mp p_{1n-1} \\ 0, & 1, & -p_{21}, & p_{22}, & \dots \mp p_{2n-1} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ 1, & -p_1, & p_2, & -p_3, & \dots \pm p_n \end{vmatrix}.$$

(The first of which is equal to

$$(1 + H_{r-n})\,\zeta^{\frac{1}{2}}(x_1, x_2, \dots x_{n+1})$$

where H_p is the sum of the homogeneous powers and products of $x_1, x_2, \dots x_{n+1}$ of order p), we obtain, by aid of the last paragraph,

$$(-)^{\frac{1}{2}\{n(n+1)\}}\zeta^{\frac{1}{2}}(x_1, x_2, \dots x_{n+1})\,\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \dots \alpha_n)\,(1 + H_{r-n}) = D.P,$$

$$+ \begin{vmatrix} \frac{f(x_1)}{x_1 - \alpha_1}, & \frac{f(x_1)}{x_1 - \alpha_2}, & \dots & \frac{f(x_1)}{x_1 - \alpha_n}, & x_1^r \\ \frac{f(x_2)}{x_2 - \alpha_1}, & \frac{f(x_2)}{x_2 - \alpha_2}, & \dots & \frac{f(x_1)}{x_2 - \alpha_n}, & x_2^r \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{f(x_{n+1})}{x_{n+1} - \alpha_1}, & \frac{f(x_{n+1})}{x_{n+1} - \alpha_2}, & \dots & \frac{f(x_{n+1})}{x_{n+1} - \alpha_n}, & x_{n+1}^r \end{vmatrix},$$

or if we cancel equal terms on each side and divide by P,

$$\begin{vmatrix} \frac{1}{x_1-\alpha_1}, & \frac{1}{x_1-\alpha_2}, \cdots & \frac{1}{x_1-\alpha_n}, & \frac{x_1^r}{f(x_1)} \\ \frac{1}{x_2-\alpha_1}, & \frac{1}{x_2-\alpha_2}, \cdots & \frac{1}{x_2-\alpha_n}, & \frac{x_2^r}{f(x_2)} \\ \cdots\cdots\cdots\

Now the coefficient of $-p_1$ in this determinant is $\Delta . P$ where the letters have the same meaning as in the first paragraph.

Hence
$$\begin{vmatrix} \frac{f(x_1)}{x_1-\alpha_1}, & \frac{f(x_1)}{x_1-\alpha_2}, & \cdots \frac{f(x_1)}{x_1-\alpha_n}, & f(x_1) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{f(x_n)}{x_n-\alpha_1}, & \frac{f(x_n)}{x_n-\alpha_2}, & \cdots \frac{f(x_n)}{x_n-\alpha_n}, & f(x_n) \\ 1 & , 1 & , \cdots & 1 \end{vmatrix}$$
$$= -\zeta^{\frac{1}{2}}(x_1, x_2, \ldots x_n)\,\zeta^{\frac{1}{2}}(\alpha_1, \ldots \alpha_n)\,(x_1+x_2+\ldots+x_n)\,(-1)^{\frac{1}{2}\{n(n-1)\}} + p_1\Delta . P.$$

Or, dividing both sides by P,
$$\begin{vmatrix} \frac{1}{x_1-\alpha_1}, & \frac{1}{x_1-\alpha_2}, & \cdots \frac{1}{x_1-\alpha_n}, & 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{x_n-\alpha_1}, & \frac{1}{x_n-\alpha_2}, & \cdots \frac{1}{x_n-\alpha_n}, & 1 \\ 1 & , 1 & , \cdots & 1 \end{vmatrix}$$
$$= \Delta\,(\alpha_1+\alpha_2+\ldots+\alpha_n-x_1-x_2\ldots-x_n).$$

5. If instead of the first determinant in the preceding paragraph we use the determinant
$$\begin{vmatrix} x_1^{\,r}+x_1^{\,n}, & x_1^{\,n-1}, & x_1^{\,n-2}, \ldots x_1, & 1 \\ x_2^{\,r}+x_2^{\,n}, & x_2^{\,n-1}, & x_2^{\,n-2}, \ldots x_2, & 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n^{\,r}+x_n^{\,n}, & x_n^{\,n-1}, & x_n^{\,n-2}, \ldots x_n, & 1 \\ 0 & , 1 , & 0 , \ldots 0 , & 0 \end{vmatrix},$$
which is equal to
$$(-1)^{n-1}\,\zeta^{\frac{1}{2}}(x_1, x_2, \ldots x_n)\,(H_1 + H_{r-n+1}),$$
we get in precisely the same way as before
$$\begin{vmatrix} \frac{1}{x_1-\alpha_1}, & \frac{1}{x_2-\alpha_2}, & \cdots \frac{1}{x_1-\alpha_n}, & \frac{x_1^{\,r}}{f(x_1)} \\ \frac{1}{x_2-\alpha_1}, & \frac{1}{x_2-\alpha_2}, & \cdots \frac{1}{x_2-\alpha_n}, & \frac{x_2^{\,r}}{f(x_2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{x_n-\alpha_1}, & \frac{1}{x_n-\alpha_2}, & \cdots \frac{1}{x_n-\alpha_n}, & \frac{x_n^{\,r}}{f(x_n)} \\ 1 & , 1 & , \cdots & 1 \end{vmatrix} = -\Delta H_{r-n+1}.$$

When H_p stands for the sum of the homogeneous powers and products of $x_1, x_2, \ldots x_n$ of order p, and Δ has the same value as in the first paragraph.

6. The determinant of order $2n$

$$\begin{vmatrix} \frac{1}{x_1-\alpha_1}, & \frac{1}{x_1-\alpha_2}, & \cdots \frac{1}{x_1-\alpha_{2n}} \\ \frac{1}{y_1-\alpha_1}, & \frac{1}{y_1-\alpha_2}, & \cdots \frac{1}{y_1-\alpha_{2n}} \\ \frac{1}{x_2-\alpha_1}, & \frac{1}{x_2-\alpha_2}, & \cdots \frac{1}{x_2-\alpha_{2n}} \\ \frac{1}{y_2-\alpha_1}, & \frac{1}{y_2-\alpha_2}, & \cdots \frac{1}{y_2-\alpha_{2n}} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{x_n-\alpha_1}, & \frac{1}{x_n-\alpha_2}, & \cdots \frac{1}{x_n-\alpha_{2n}} \\ \frac{1}{y_n-\alpha_1}, & \frac{1}{y_n-\alpha_2}, & \cdots \frac{1}{y_n-\alpha_{2n}} \end{vmatrix}$$

being of the form of that in paragraph 1 is equal to the product

$$\frac{(-1)^{n(2n-1)}\,\zeta^{\frac{1}{2}}(x_1, x_2, \ldots x_n)\,\zeta^{\frac{1}{2}}(y_1, y_2, \ldots y_n)\,\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \ldots \alpha_{2n})}{f(x_1)\ldots f(x_n)\,f(y_1)\ldots f(y_n)}$$

$$\times (y_1-x_2)(y_1-x_3)(y_1-x_4)\ldots(y_1-x_n)$$
$$(y_2-x_3)(y_2-x_4)\ldots(y_2-x_n)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(y_{n-1}-x_n)$$
$$\times (x_1-y_1)(x_1-y_2)(x_1-y_3)\ldots(x_1-y_n)$$
$$(x_2-y_2)(x_2-y_3)\ldots(x_2-y_n)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(x_n-y_n).$$

Now in the determinant subtract the $(r+1)^{\text{th}}$ row from the $(r+2)^{\text{th}}$ the elements of the latter row then become

$$\frac{x_r-y_r}{(x_r-\alpha_1)(y_r-\alpha_1)}, \quad \frac{x_r-y_r}{(x_r-\alpha_2)(y_r-\alpha_2)}, \quad \cdots \frac{x_r-y_r}{(x_r-\alpha_{2n})(y_r-\alpha_{2n})},$$

and thus x_r-y_r divides out of both sides of the equality.

Do this for all values of r from 1 to n and then let $x_i = y_i$ for all values of i from 1 to n, and we obtain the equality

$$\begin{vmatrix} \frac{1}{x_1-\alpha_1}, & \frac{1}{x_1-\alpha_2}, & \cdots & \frac{1}{x_1-\alpha_{2n}} \\ \frac{1}{(x_1-\alpha_1)^2}, & \frac{1}{(x_1-\alpha_2)^2}, & \cdots & \frac{1}{(x_1-\alpha_{2n})^2} \\ \frac{1}{x_2-\alpha_1}, & \frac{1}{x_2-\alpha_2}, & \cdots & \frac{1}{x_2-\alpha_{2n}} \\ \frac{1}{(x_2-\alpha_1)^2}, & \frac{1}{(x_2-\alpha_2)^2}, & \cdots & \frac{1}{(x_2-\alpha_{2n})^2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{x_n-\alpha_1}, & \frac{1}{x_n-\alpha_2}, & \cdots & \frac{1}{x_n-\alpha_{2n}} \\ \frac{1}{(x_n-\alpha_1)^2}, & \frac{1}{(x_n-\alpha_2)^2}, & \cdots & \frac{1}{(x_n-\alpha_{2n})^2} \end{vmatrix}$$

$$= \frac{(-1)^n \{\zeta^{\frac{1}{2}}(x_1, x_2, \ldots x_n)\}^4 \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2, \ldots \alpha_{2n})}{\{f(x_1)\ldots f(x_n)\}^2}.$$

This theorem, which was given without proof by Brioschi (*Sullo sviluppo di un Determinante*, Annali di Mat. (N. S) I. p. 9, [1857]), is clearly capable of generalisation. The general result may be stated as follows. If a determinant of order nr is formed by n groups of r rows of which the i^{th} group is

$$(x_i-\alpha_1)^{-1}, \ (x_i-\alpha_2)^{-1}, \ \ldots (x_i-\alpha_{nr})^{-1},$$
$$(x_i-\alpha_1)^{-2}, \ (x_i-\alpha_2)^{-2}, \ \ldots (x_i-\alpha_{nr})^{-1},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(x_i-\alpha_1)^{-r}, \ (x_i-\alpha_2)^{-r}, \ \ldots (x_i-\alpha_{nr})^{-r}.$$

Then the value of the determinant so found is

$$\frac{\{\zeta^{\frac{1}{2}}(x_1, x_2, \ldots x_n)\}^{r^2} \zeta^{\frac{1}{2}}(\alpha_{nr}, \alpha_{nr-1}, \ldots \alpha_1)}{\{f(x_1), \ldots f(x_n)\}^r},$$

where $$f(x) = (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{nr}).$$

The proof of this theorem would occupy a considerable space, but offers no difficulty if the method used for the case $r=2$ be employed.

TWO CONSTRUCTIONS FOR DRAWING SPHERES TO TOUCH FOUR GIVEN SPHERES, AND AN ASSOCIATED THEOREM.

By *G. F. Walker, M.A.*, Fellow of Queens' College, Cambridge.

To draw a sphere to touch four given spheres.

The following is the construction:

Take the poles with respect to the four spheres of any one of the planes each of which contains six of the centres of similitude of the spheres taken two and two (excluding a plane of three centres). Join these poles to the radical centre of the four spheres, and the joining lines will cut the respective spheres in the points of contact of a pair of conjugate spheres touching the four.

Let the four spheres be called A, B, C, and D, and let a, b, c, d be the points of contact with them respectively of a sphere touching the four. To fix the ideas suppose the contacts all external. Then the plane of six centres to take will be the plane through the six external centres of similitude. The lines bc, ca, ab must pass through the external centre of similitude of the pairs of spheres BC, CA, AB respectively, and therefore meet the plane of centres of A, B, C in its line of intersection with the plane of six centres of similitude which we shall call the plane S. Let this line be called s_1. The plane abc will evidently cut the spheres A, B, C, $abcd$ in four circles, the last of which will touch the other three, the circles a, b, c, abc say. Let o_1 be the radical centre of a, b, c and p_1, p_2, and p_3 the poles of s_1 with respect to a, b, and c, and P_1, P_2, P_3, P_4 the poles of S with respect to A, B, C, and D respectively.

Then, by the well-known construction for drawing a circle to touch three given circles, the lines o_1p_1, o_1p_2, o_1p_3 will pass through a, b, and c respectively, since s_1 contains the three external centres of similitude of a, b, and c.

The point o_1 will evidently lie on the radical axis of A, B, and C, and the points p_1, p_2, and p_3 on the reciprocal lines of s_1 with respect to A, B, and C, and these reciprocal lines will also pass through P_1, P_2, and P_3 respectively and be perpendicular to the plane of centres of A, B, C, and will therefore be parallel to the radical axis of A, B, and C.

Hence, the line P_1a will intersect the radical axis of A, B, and C.

Similarly it will intersect the radical axes of A, C, D, and of A, D, B, and since these lines are not in one plane and meet in one point (the radical centre of the four spheres) the line P_1a must pass through this radical centre. Similarly for the others. Hence the construction.

If we had supposed all the contacts internal we should have had the same plane S; and, therefore, the same construction; and, therefore, OaP_1 etc. meet their respective spheres again in the points of contact of another sphere touching the four.

Let these points be a', b', c', d' respectively. Since the line aa' contains the points P_1 and O (the radical centre) the tangent planes to A at a and a' intersect in a line which lies in the plane S and in the polar plane of O with respect to A.

But this polar plane is the plane of intersection with A of the sphere, centre O, cutting A at right angles, and this sphere also cuts B, C, and D at right angles since O is the radical centre.

Hence, we have also the following construction for drawing spheres to touch four given spheres.

Draw the sphere which cuts the four given spheres at right angles, and let its planes of intersection with the four spheres cut any one of the planes, each of which contains six centres of similitude of the four spheres taken two and two (a plane of three centres being excluded), in four lines.

Through these lines draw tangent planes to the respective spheres, then their points of contact will be the points of contact of two conjugate spheres touching the four.

Since there are eight planes such as S and each gives two spheres there will be 16 spheres in all.

The following theorem, the latter part of which has been proved incidentally above, may be noticed.

If a sphere touch three given spheres the locus of its centre consists, in general, of eight plane conics, and the locus of its point of contact with any of the spheres consists of four small circles, the planes of all twelve small circles passing through the same line, the radical axis of the three spheres.

CONSTRUIRE LES AXES D'UNE ELLIPSE DONT ON CONNAIT DEUX DIAMETRES CONJUGUES.

By Prof. *A. Mannheim.*

Ce problême a été souvent résolu. Je l'ai déja traité dans un article qui a paru en 1878, dans les nouvelles Annales de Mathématiques. Je le reprends aujourd' hui pour en apporter de nouvelles solutions.

Voici, pour la première solution, ce que je suppose connu.

Sur les ordonnées telles que fp (fig. 32), des points d'une circonférence (o), on construit des triangles, tels que fph, semblables à un triangle donné: le lieu des points h ainsi déterminés est une ellipse dont le centre est o. La tangente en h à cette courbe et la tangente en f à la circonférence (o) se rencontrent en un même point t de on. D'après cela, au point m, qui correspond au point l extrémité du rayon perpendiculaire à on, la tangente est parallèle à la tangente en l à la circonférence (o), c'est à dire qu'elle est parallèle à on. Nous voyons ainsi que *on et om sont deux demi diamètres conjugués de l'ellipse lieu des points h.*

Nous appellerons (m) cette ellipse et nous nous proposons de construire ses axes en supposant donnés ces deux demi diamètres conjugués om et on.

Si le point h était l'un des sommets de (m) l'angle oht serait droit et les points f, h seraient sur la circonférence décrite sur ot comme diamètre. Cette remarque permet tout de suite de construire une figure semblable à la figure, telle que $ofht$, qu'on obtiendrait si h était un sommet de (m).

Construisons cette figure. Par les points l et m faisons passer une circonférence dont le centre soit sur no. Cette circonférence passe par l', symétrique de l, et elle coupe la droite no aux points o' et t'. On voit tout de suite que la figure o', l, m, t', est semblable à la figure que nous nous proposons de trouver et alors *mo' et mt' sont les directions des axes de* (m).

Cette construction montre que ces axes sont parallèles aux bissectrices des angles formés par les droites ml, ml' que l'on obtient en joignant le point m aux extrémités des segments ol, ol' perpendiculaires à on et égaux à ce demi-diamètre.

Du point o menons la parallèle $o\lambda$ à $o'l$. Cette droite rencontre la circonférence (o) au point λ qui correspond au sommet μ de l'ellipse m. Nous allons simplifier cette construction en la modifiant.

On a $$\frac{o'l}{o\lambda} = \frac{o'm}{o\mu},$$

comme $o\lambda$ est égal à ol, on tire de là

$$o\mu = o'm\frac{ol}{o'l} = o'm\cos olo' = o'm\cos l'mo' = mv,$$

le point v étant le pied de la perpendiculaire abaissée du point o' sur ml'. Le segment mv est donc égal au demi-axe $o\mu$. De même on aurait l'autre demi-axe en projetant mt' sur ml ou sur ml'.

Ainsi : *Les longueurs des demi-axes de* (m) *sont les projections des segments* mo', mt' *sur les côtés de l'angle* lml'.

En partant de cette solution, nous allons retrouver une construction connue.

Le point o' est sur la circonférence circonscrite au triangle lml'. Les pieds des perpendiculaires abaissées de ce point sur les côtés du triangle lml' sont alors en ligne droite et comme le point o' est sur la bissectrice mo' de l'angle $l'ml$, cette droite est perpendiculaire à mo'. Ainsi vo est perpendiculaire à mo', par suite parallèle à mt'.

Menons la droite qui joint le point o au milieu j du segment lm ; cette droite oj est parallèle à mv. Prolongeons mt' jusqu'à sa rencontre en a avec oj, nous obtenons le parallélogramme $ovma$, et nous voyons que oa *est égal à* mv, c'est-à-dire à l'un des demi-axes de (m). De la même manière on trouve que ob est égal à l'autre demi-axe.

Puisque oj est parallèle à mv le triangle bmj est isoscèle et l'on a $jb = jm$. De même $ja = jm$ et nous retrouvons cette construction.

Perpendiculairement à on, *on élève le segment* ol *égal à* on. *On joint le point* o *au milieu* j *du segment* lm *et l'on porte sur cette droite les segments* jb *et* ja *égaux au segment* jm. *Les droites* ma, mb *sont les directions des axes de l'ellipes* (m) *et* oa, ob *sont les demi-axes de cette courbe.*

Nous allons maintenant montrer comment on peut arriver à cette solution sachant seulement que le lieu des points tels que h est une ellipse dont om et on sont deux demi-diamètres conjugués. Je ne supposerai plus connue la construction de la tangente ht au point quelconque h de cette courbe.

Faisons tourner (fig. 33) le quadrilatère $ofph$ autour du point o jusqu' à ce que le point f vienne en l. Le point p vient alors en un point p' de la circonférence décrite sur ol comme diamètre. Le triangle fph vient en $lp'h'$. A chacun des points, tels que h, de l'ellipse (m) correspond un point h'.

Le lieu des points h' est le lieu des sommets des triangles, tels que $lp'h'$, semblables au triangle lom et qui sont construits sur les cordes telles que lp', de la circonférence $op'l$. Ce lieu est alors une circonférence de cercle qui a pour diamètre le segment lm, qui fait avec lo un angle égal à l'angle $p'lh'$ et qui est tel que $\frac{lm}{lo} = \frac{lh'}{lp'}$.

Cette circonférence rencontre la circonférence $op'l$ au point q où om coupe cette dernière circonférence. Le côté $p'h'$ passe par ce point q, parce que l'angle $h'p'l$, est égal à l'angle qol. De la construction du point h', il résulte que la distance du point o à h' est égal à la distance du point o à h. Si le point h était l'extrémité du grand axe, le demi-diamètre oh serait le plus grand des demi-diamètres de (m) et correspondrait alors au point a où le diamètre oj rencontre la circonférence lieu des points h'.

Connaissant oa, nous allons chercher le demi-diamètre de (m) qui lui correspond de la même façon que oh' correspond à oh. Joignons le point a au point q, cette droite donne sur la circonférence $op'l$ le point p'' analogue au point p'. Nous devons maintenant faire tourner le triangle $op''a$ autour du point o jusqu à ce que op'' vienne sur on: la droite oa viendra alors coincider avec le demi-grand axe de (m).

L'angle dont tourne le triangle $op''a$ est nop'' qui est égal à l'angle olp''.

Mais $olp'' = olm + mlp'' = p''la + mlp'' = mla = laj$,

par suite, la droite oa vient se placer parallèlement à al. Ainsi le demi grand axe de (m) est parallèle à al et sa longueur est oa. On voit de même que le demi petit axe de cette courbe est égal à ob et parallèle à bl; nous avons donc retrouvé la construction précédente.

ON A SIX-POINT CIRCLE CONNECTED WITH A TRIANGLE.

By *H. M. Taylor, M.A.*

If from the angular points A, B, C of a triangle perpendiculars AD, BE, CF be drawn to the opposite sides BC, CA, AB, and also two sets of perpendiculars Fc, Da, Eb and $E\beta$, $F\gamma$, $D\alpha$ be drawn to the same sides in order; then the six points a, γ, b, α, c, β lie on a circle.

Because the angles $E\beta D$, EaD are right angles, $Ea\beta D$ is a circle; and because the angles AEB, ADB are right angles, $AEDB$ is a circle (fig. 34). Therefore

$\angle Da\beta = \angle DEB =$ alternate angle $ADE = \angle ABE$,

and BE, Da are parallel; therefore $a\beta$ is parallel to AB.

Similarly $b\gamma$ is parallel to BC and $c\alpha$ is parallel to CA.

The triangles $Ab\gamma$, αBc, $a\beta C$ are all similar to the whole triangle ABC.

Again, because the angles FcC, $F\gamma C$ are right angles, $F\gamma Cc$ is a circle; therefore

$$\angle c\gamma C = \angle cFC = \angle B;$$

therefore $$\angle \gamma cC = \angle A,$$

and the triangle $C\gamma c$ is similar to the triangle CBA.

In the same way it may be shown that the triangles $Aa\alpha$, βBb are similar to the triangle ABC.

Now, because the triangles $Ca\beta$, $C c\gamma$ are similar to the triangle CAB, and therefore to one another, $c\beta a\gamma$ is a circle; and because the triangles $Aa\alpha$, ABC are similar,

$$\angle Aa\alpha = \angle B,$$

but $$\angle \beta aC = \angle A;$$

therefore $$\angle \alpha a\beta = \angle C = \angle \alpha cB;$$

therefore $\alpha c\beta a$ is a circle; but it has already been shown that $c\beta a\gamma$ is a circle.

Therefore the five points $\alpha c\beta a\gamma$ lie on a circle; *i.e.* the circle $\alpha\beta\gamma$ passes through the points a and c, and therefore, by symmetry, through the point b also; or, in other words, $abc\alpha\beta\gamma$ lie on a circle.

Now the angle $\beta\gamma\alpha = \angle \beta a\alpha$ in the same segment of circle $abc\alpha\beta\gamma = \angle C$.

Similarly, $\angle \gamma\alpha\beta = \angle A$ and $\angle \alpha\beta\gamma = \angle B$.

Therefore the triangle $\alpha\beta\gamma$ is similar to the triangle ABC.

Similarly, the triangle abc is similar to the triangle ABC.

It appears therefore that the triangles $\alpha\beta\gamma$, abc are similar to each other, and, because they are in the same circle, they are equal in all respects; therefore

$$ab = \alpha\beta, \qquad bc = \beta\gamma, \qquad ca = \gamma\alpha;$$

and it follows also, because the points abc, $\alpha\beta\gamma$ occur in the same order in passing along the circumference of the circle, that

$$a\alpha = b\beta = c\gamma.$$

Also, if $\beta\varepsilon$ be drawn perpendicular to AB,

$$\beta\varepsilon = B\beta \sin B, \quad B\beta = BE \sin C, \quad BE = BA \sin A;$$

therefore $$\beta\varepsilon = c \sin A \sin B \sin C.$$

Again, $\alpha\varepsilon = D\beta \cos B$, $D\beta = AE \cos C$, $AE = AB \cos A$;

therefore $$\alpha\varepsilon = c \cos A \cos B \cos C;$$

therefore $$\alpha\beta^2 = \alpha\varepsilon^2 + \beta\varepsilon^2$$
$$= c^2 (\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C).$$

It appears therefore that the linear relations of either of the equal triangles abc, $\alpha\beta\gamma$ to the triangle ABC is

$$\sqrt{(\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C)} : 1.$$

Again, because $a\alpha$ is a chord of the circle $AaD\alpha$, which subtends an angle A at the circumference, and AD is the diameter

$$a\alpha = AD \sin A = AB \sin B \sin A$$
$$= \frac{c}{\sin C} \sin A \sin B \sin C$$
$$= 2R \sin A \sin B \sin C = 2R' \sin \phi,$$

if ϕ be the angle subtended by $a\alpha$ at any of the points b, β, c, γ, and R, R' be the radii of the circumscribed circles of the triangles ABC, $\alpha\beta\gamma$ respectively. But

$$R'^2 = R^2 (\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C),$$

and $$R' \sin \phi = R \sin A \sin B \sin C;$$

therefore $$R' \cos \phi = \pm R \cos A \cos B \cos C,$$

and therefore $$\tan \phi = \pm \tan A \tan B \tan C.$$

The last result might have been deduced at once from the expressions for $\beta\varepsilon$ and $\alpha\varepsilon$.

NOTES ON CENTRAL FORCES.

By *Arthur Hill Curtis, LL.D., D.Sc.*

I. *Moveable Orbits.*

When two orbits are described round the same centre of force under the restriction that the distances of the moving particles from this centre shall be equal at each instant of time the following conditions result:

(*a*) The angular velocities of the particles round the centre of force at each instant are in a constant ratio.

(*b*) The difference of the two central forces acting on the particles at corresponding points of the two orbits varies inversely as the cube of the distance.

(*c*) The difference between the squares of the velocities of the particles at the corresponding points of the two orbits varies inversely as the square of the distance.

To prove (*a*):—The centre of force being taken as origin, let r_1, θ_1, at any instant of time, be the polar coordinates of the particle which describes one orbit, for the purpose of reference denoted by A, while p_1 denotes the perpendicular from the origin on the direction of motion at the same instant, and h_1 double the area described round the origin in the unit of time; let r_2, θ_2, p_2, h_2 denote the corresponding quantities

for the other orbit, denoted by B; we then have the following conditions:

$$r_1^2 d\theta_1 = h_1 dt, \quad r_2^2 d\theta_2 = h_2 dt, \quad r_1 = r_2,$$

hence $\frac{d\theta_1}{d\theta_2} = \frac{h_1}{h_2}$, which proves ($a$).

To prove (b):

$$F_1 dr_1 = \tfrac{1}{2} h_1^2 \, d\left(\frac{1}{p_1^2}\right) = \tfrac{1}{2} h_1^2 \, d\left(\frac{1}{r_1^2} + \frac{dr_1^2}{r_1^4 d\theta_1^2}\right)$$

$$= \tfrac{1}{2} h_1^2 \, d\left(\frac{1}{r_1^2} + \frac{dr_1^2}{h_1^2 dt^2}\right) = h_1^2 \left(-\frac{1}{r_1^3} + \frac{d^2 r_1}{h_1^2 dt^2}\right) dr_1,$$

or
$$F_1 = -\frac{h_1^2}{r_1^3} + \frac{d^2 r_1}{dt^2};$$

similarly
$$F_2 = -\frac{h_2^2}{r_2^3} + \frac{d^2 r_2}{dt^2},$$

but $r_1 = r_2$ ($= r$ suppose) *all through the motion;* therefore

$$\frac{dr_1}{dt} = \frac{dr_2}{dt} \text{ and } \frac{d^2 r_1}{dt^2} = \frac{d^2 r_2}{dt^2},$$

therefore $F_2 - F_1 = -(h_2^2 - h_1^2)\frac{1}{r^3}$, which proves ($b$).

The force denoted by $F_2 - F_1$ is directed towards the centre of force if $h_2 > h_1$, and from it if $h_1 > h_2$.

To prove (c):—

$$v_1^2 = \frac{h_1^2}{p_1} = h_1^2 \left(\frac{1}{r_1^2} + \frac{dr_1^2}{r_1^4 d\theta_1^2}\right)$$

$$= h_1^2 \left(\frac{1}{r_1^2} + \frac{dr_1^2}{h_1^2 dt^2}\right) = \frac{h_1^2}{r_1^2} + \frac{dr_1^2}{dt^2};$$

similarly
$$v_2^2 = \frac{h_2^2}{r_2^2} + \frac{dr_2^2}{dt^2},$$

and, as $r_1 = r_2 = r$ all through the motion,

$$\frac{dr_1}{dt} = \frac{dr_2}{dt} \text{ and } v_2^2 - v_1^2 = \frac{h_2^2 - h_1^2}{r^2},$$

which proves (c).

Either A or B may be called the *moveable* orbit of the other (Newton's *Principia Mathematica*, Book I. Section IX.); that is to say, supposing $h_2 > h_1$, the position of the particle in the orbit B may be deduced from the corresponding position of the particle in the orbit A by turning the radius vector of the latter in the same direction as the angular

motion of the particle through an angle $= \left(\frac{h_2}{h_1} - 1\right)\theta_1$, while, conversely, the position of the particle in the orbit A may be deduced from that in the orbit B by turning the radius vector in the direction opposite to the angular motion of the particle through the same angle, or co-directionally through the angle $\left(\frac{h_1}{h_2} - 1\right)\theta_2$, which is $= -\left(\frac{h_2}{h_1} - 1\right)\theta_1$.

From the condition (a) it immediately follows that $\theta_2 = \frac{h_2}{h_1}(\theta_1 + \alpha)$; the constant α may be omitted, as it merely leaves arbitrary the azimuth of the orbit B in the plane of motion, so that we may write $\theta_2 = \frac{h_2}{h_1}\theta_1$. As r is the same at every instant in the two orbits, it follows that the maximum and minimum values in the two orbits correspond, and therefore, if the angle between two consecutive apsides in the orbit A be denoted by γ, the corresponding angle, in the orbit B, will be denoted by $\frac{h_2}{h_1}\gamma$.

From the condition (c) it follows that, when the central force corresponding to the orbit A is expressed by a series of the form $-\sum \frac{\mu}{r^n}$, where $n > 1$, if the velocity of the moving particle at any one, and therefore at every, point in its orbit be equal to that required in being drawn in from (or pushed out to) rest at infinity, the same is true of the orbit B; for the former condition is expressed by

$$v_1^2 = \sum \frac{2\mu}{(n-1)\,r^{n-1}},$$

and therefore by (c),

$$v_2^2 = \sum \frac{2\mu}{(n-1)\,r^{n-1}} + \frac{h_2^2 - h_1^2}{r^2},$$

which expresses the condition that the same should be true of the orbit corresponding to

$$F = -\sum \frac{\mu_n}{r^n} - \frac{(h_2^2 - h_1^2)}{r^3},$$

that is of the orbit B.

When the orbit corresponding to the central force F is known, that corresponding to the central force $F - \frac{\nu}{r^3}$ is also known, for, if the orbit A in the former case be defined by

$r_1 = f(\theta_1)$, the orbit B in the latter will be $r_2 = f\left(\frac{h_1}{h_2}\theta_2\right)$, where $\nu = h_2^2 - h_1^2$, and h_1 is known in terms of the several constants denoted by μ_n and those contained in the function f.

The principle here established may be applied either for the determination of the law of force in an orbit which may be derived from another orbit, in which the law of force (supposed central) is known, by altering θ in a constant ratio, or for the determination of the orbit described under a central force which can be obtained from the law of force (supposed central) corresponding to a known orbit by introducing in addition a force directed towards, or from, the same centre and varying inversely as the cube of the distance. The following examples will illustrate the method:

Ex. 1. $$F_1 = \pm\frac{\mu}{r_1^2};$$

here orbit A is the focal conic

$$r_1 = \frac{a(1-e)^2}{1-e\cos(\theta_1-\beta)};\quad h_1^2 = \mu a(1-e^2),\text{ and }\gamma_1 = \pi;$$

therefore, when

$$F_2 = \pm\frac{\mu}{r_2^2} - \frac{\nu}{r_2^3},$$

the orbit B is

$$r^2 = \frac{a(1-e^2)}{1-e\cos\left(\frac{h_1}{h_2}\theta_2 - \beta\right)},$$

where $\nu = h_2^2 - h_1^2$, and therefore

$$h_2 = \sqrt{(\nu + h_1^2)} = \sqrt{\{\nu + \mu a(1-e^2)\}},$$

so that the orbit is

$$r_2 = \frac{a(1-e^2)}{1-e\cos\left\{\sqrt{\left(\frac{\mu a(1-e^2)}{\nu+\mu a(1-e^2)}\right)}\,\theta_2 - \beta\right\}}$$

(see Tait and Steele's *Dynamics*, Edition of 1878, p. 155, Ex. 22),

and $$\gamma_2 = \sqrt{\left(\frac{\nu+\mu a(1-e^2)}{\mu a(1-e^2)}\right)}\,\pi.$$

Ex. 2. $F_1 = \pm\mu r_1$; orbit A is the central conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,\text{ or }\frac{1}{r_1^2} + \frac{\cos^2(\theta_1-\beta)}{a^2} - \frac{\sin^2(\theta_1-\beta)}{b^2};$$

$h_1^2 = \mu ab$ and $\gamma_1 = \frac{1}{2}\pi$; therefore, when

$$F_2 = \pm \mu r_2 - \frac{\nu}{r^3},$$

the orbit B is

$$\frac{1}{r_2^2} = \frac{\cos^2\left\{\sqrt{\left(\frac{\mu ab}{\nu + \mu ab}\right)}\,\theta_2 - \beta\right\}}{a^2} + \frac{\sin^2\left\{\sqrt{\left(\frac{\mu ab}{\nu + \mu ab}\right)}\,\theta_2 - \beta\right\}}{b^2}$$

(proposed at B.A. Honour Examination, Queen's University, 1875, see *Queen's University Calendar;* also Tait and Steele's *Dynamics*, Edition of 1878, p. 155, Ex. 21, § 1),

and $$\gamma_2 = \sqrt{\left(\frac{\nu + \mu ab}{\mu ab}\right)}\,\tfrac{1}{2}\pi.$$

Ex. 3. $F_1 = -\frac{\mu}{r_1^n}$, where $n > 1$, with the condition that the velocity at any one, and therefore at every, point of the orbit is equal to that acquired in being drawn in from (or expended in being pushed out to) rest at infinity; the equation of the orbit A is

$$r_1^{\frac{1}{2}(n-3)} = \sqrt{\left(\frac{2\mu}{(n-1)\,h_1^2}\right)} \cos\left[\left\{\tfrac{1}{2}(n-3)\right\}(\theta_1 - \alpha)\right];$$

hence, if $F = \mp\frac{\mu}{r^n} - \frac{\nu}{r^3}$, with the same restriction as to initial velocity, the orbit B is expressed by

$$r_2^{\frac{1}{2}(n-3)} = \sqrt{\left(\frac{2\mu}{(n-1)\,h_1^2}\right)} \cos\left[\left\{\tfrac{1}{2}(n-3)\right\}(k\theta_2 - \alpha)\right],$$

where $$h_2^2 - h_1^2 = \nu \text{ and } k = \frac{h_2}{h_1} = \frac{h_2}{\sqrt{(h_2^2 - \nu)}},$$

or by

$$r_2^{\frac{1}{2}(n-3)} = \sqrt{\left(\frac{2\mu}{(n-1)\,(h_2^2 - \nu)}\right)} \cos\tfrac{1}{2}(n-3)\left(\frac{h_2}{\sqrt{(h_2^2 - \nu)}}\,\theta_2 - \alpha\right).$$

One class of cases requires special consideration, viz. when $\nu = h_2^2$, viz. in other words, when $h_1 = 0$; in this case the elementary area in the orbit A vanishes, and this orbit becomes a straight line passing through the centre of force; the orbit B in this case cannot be said to be derived from the orbit A by altering the angular velocity of the moving particle in a constant ratio, but the orbit B may still be

defined as the orbit in which the radius vector at each instant is the same as in the corresponding rectilinear path, while the angular velocity all through the motion is such as to satisfy the law of conservation of areas. The orbit B may, therefore, in this case be obtained by eliminating t between the equation which defines, in the rectilineal path A, the distance of the particle from the centre of force as a function of t, and the equation $r^2 \frac{d\theta}{dt} = h$, or, as we may now call it, h. A few examples will illustrate the method:

Ex. 1. $$F = -\frac{\mu}{r^2} - \frac{h^2}{r^3};$$

the motion in the rectilineal path A is here defined by the equation

$$\frac{d^2r}{dt^2} = -\frac{\mu}{r^2},$$

from which we obtain, by integration,

$$\frac{dr^2}{dt^2} = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right),$$

and the motion in the orbit B is defined by this equation combined with $r^2 d\theta = h dt$, and thus we obtain the equation

$$r^2 d\theta = \pm \frac{h}{\sqrt{(2\mu)}} \frac{dr}{\sqrt{\left(\frac{1}{r} - \frac{1}{a}\right)}},$$

or $$\frac{\sqrt{(2\mu)}}{2h}(\theta + \alpha) = \mp \sqrt{\left(\frac{1}{r} - \frac{1}{a}\right)};$$

or, putting $$\frac{\sqrt{(2\mu)}}{2h} = \sqrt{\lambda}, \quad \frac{1}{r} - \frac{1}{a} = \lambda(\theta + \alpha)^2,$$

or $$r = \frac{a}{1 + a\lambda(\theta + \alpha)^2},$$

the equation of the orbit B (for a particular case, see Tait and Steele's *Dynamics*, p. 155, Ex. 22, §2).

Ex. 2. $$F = -\mu r - \frac{h^2}{r^3};$$

the motion in the rectilineal path A is here defined by the equation

$$\frac{d^2r}{dt^2} = -\mu r,$$

from which we obtain, by integration,

$$r = L\cos\{\sqrt{(\mu)}t + \alpha\},$$

and the motion in orbit B is defined by this equation combined with the condition $r^2 d\theta = h dt$, and thus we obtain

$$d\theta = \frac{h}{r^2}dt = \frac{h dt}{L^2\cos^2\{\sqrt{(\mu)}t+\alpha\}} = \frac{h}{L^2\sqrt{\mu}}\, d\tan\{\sqrt{(\mu)}t+\alpha\},$$

from which we derive

$$\theta + \beta = \frac{h}{L^2\sqrt{\mu}}\tan\{\sqrt{(\mu)}t+\alpha\} = \frac{h}{L^2\sqrt{\mu}}\sqrt{\left(\frac{L^2}{r^2}-1\right)},$$

or

$$r^2 = \frac{L^2}{1+\dfrac{L^2\sqrt{\mu}}{h}(\theta+\beta)^2}$$

(proposed at B.A. Honour Examination, Queen's University, 1875. For a particular case, see Tait and Steele's *Dynamics*, p. 155, Ex. 21, §2).

Ex. 3. $$F = -\frac{\mu}{r^n} - \frac{h^2}{r^3},$$

where $n > 1$, with the condition that the velocity in path A, and therefore in path B, as already proved, page 181, is equal to that acquired in being pulled in from infinity; the motion in the rectilineal path A is here defined by the equation

$$\frac{d^2r}{dt^2} = -\frac{\mu}{r^n};$$

or, integrating,

$$\frac{dr^2}{dt^2} - \frac{2\mu}{(n-1)\,r^{n-1}} - C,$$

but from the condition of the question, C vanishes; therefore

$$\frac{dr^2}{dt^2} = \frac{2\mu}{(n-1)\,r^{n-1}};$$

therefore

$$r^{\frac{1}{2}(n-1)}dr = \pm\sqrt{\left(\frac{2\mu}{n-1}\right)}\,dt,$$

or in orbit B

$$r^{\frac{1}{2}(n-1)}dr = \pm\sqrt{\left(\frac{2\mu}{n-1}\right)}\,\frac{r^2 d\theta}{h};$$

therefore

$$r^{\frac{1}{2}(n-5)}dr = \sqrt{\left\{\frac{2\mu}{(n-1)\,h^2}\right\}}\,d\theta,$$

or $$r^{\frac{1}{2}(n-3)} = \pm\tfrac{1}{2}(n-3)\left\{\frac{2\mu}{(n-1)\,h^2}\right\}^{\frac{1}{2}}(\theta+\alpha).$$

(For *particular case*, viz. where $n=5$, see Tait and Steele's *Dynamics*, p. 154, Exs. 16 and 17.)

II. *The Law of Force, supposed central, corresponding to an orbit geometrically related to the centre of force and a given curve:*

(i) When the orbit is the *inverse* of a known curve with regard to the centre of force.

Let r, p; R, P be the radius vector and perpendicular on tangent for any point on the orbit, and for the curve respectively, at corresponding points, then $rR = K^2$, and

$$\frac{r}{p} = \frac{R}{P} = \frac{K^2}{Pr};$$

therefore $\dfrac{r^2}{p} = \dfrac{K^2}{P}$, and, if the equation of the given curve be written $P = f(R)$, then

$$\frac{K^2 p}{r^2} = f\left(\frac{K^2}{r}\right);$$

this equation establishes a relation between r and p, from which, by the equation

$$F dr = (\tfrac{1}{2}h^2)\,d\left(\frac{1}{p^2}\right),$$

F may be found as a function of r, for example:

(1) Let the orbit be the inverse of an ellipse whose focus is at the centre of force. Here by a well-known property of the ellipse

$$\frac{1}{P^2} = \frac{2a-R}{b^2R} = \frac{2a}{b^2}\,\frac{1}{R} - \frac{1}{b^2};$$

therefore $$\frac{r^4}{p^2} = K^4\left(\frac{2a}{b^2}\,\frac{r}{K^2} - \frac{1}{b^2}\right),$$

or $$\frac{1}{p^2} = K^4\left(\frac{2a}{K^2b^2}\,\frac{1}{r^3} - \frac{1}{b^2r^4}\right) = \frac{2aK^2}{b^2}\,\frac{1}{r^3} - \frac{K^4}{b^2r^4};$$

therefore

$$Fdr = (\tfrac{1}{2}h^2)\, d\left(\frac{1}{p^2}\right) = \left(-\frac{3ah^2K^2}{b^2}\frac{1}{r^4} + \frac{2h^2K^4}{b^2}\frac{1}{r^5}\right) dr,$$

and therefore

$$F = -\frac{3ah^2K^2}{b^2}\frac{1}{r^4} + \frac{2h^2K^2}{b^2}\frac{1}{r^5}.$$

(2) Let the orbit be the inverse of a central conic whose centre is at the centre of force.

Here we have

$$a^2b^2 = P^2b'^2 = P^2(a^2 + b^2 - R^2),$$

therefore

$$a^2b^2 = \frac{K^4p^2}{r^4}\left(a^2 + b^2 - \frac{K^4}{r^2}\right);$$

therefore

$$\frac{1}{p^2} = \frac{K^4(a^2+b^2)}{a^2b^2}\frac{1}{r^4} - \frac{K^8}{a^2b^2r^6};$$

therefore

$$Fdr = \tfrac{1}{2}h^2\, d\left(\frac{1}{p^2}\right) = \left\{-\frac{2h^2K^4(a^2+b^2)}{a^2b^2}\frac{1}{r^5} + \frac{3h^2K^8}{a^2b^2}\frac{1}{r^7}\right\} dr;$$

and therefore

$$F = -\frac{2h^2K^4(a^2+b^2)}{a^2b^2}\frac{1}{r^5} + \frac{3h^2K^8}{a^2b^2}\frac{1}{r^7}.$$

Cor. (i). As a particular case, when the conic is an equilateral hyperbola $a^2 + b^2 = 0$, the orbit is the lemniscata of Bernouilli, and the law of force is given by the equation

$$F = -\frac{3h^2K^8}{a^4}\frac{1}{r^7},$$

(see Tait and Steele's *Dynamics*, p. 146, Ex. 4).

(ii) When the orbit is the reciprocal of a known curve with regard to the centre of force. Assuming the same notation as in case (i) we now have $rP = Rp = K^2$; therefore, if the equation of the known curve be $R = f(P)$, we obtain

$$\frac{K^2}{p} = f\left(\frac{K^2}{r}\right);$$

and, combining this relation with the equation

$$Fdr = \tfrac{1}{2}h^2 d\left(\frac{1}{p^2}\right);$$

F may be obtained as a function of r, for example:

(1) Let the orbit be the reciprocal of a conic whose focus is at the centre of force.

Here $$\frac{1}{P^2} = \frac{2a - R}{b^2 R} = \frac{2a}{b^2}\frac{1}{R} - \frac{1}{b^2};$$

therefore $$\frac{r^2}{K^4} = \frac{2a}{b^2}\frac{p}{K^2} - \frac{1}{b^2};$$

therefore $$F dr = \tfrac{1}{2}h^2 d\left(\frac{1}{p^2}\right) = -\frac{8a^2h^2K^4 r dr}{b^4\left(r^2 + \frac{K^4}{b^2}\right)^3},$$

and, therefore, $$F = -\frac{8a^2h^2K^4 r}{b^2\left(r^2 + \frac{K^4}{b^2}\right)^3},$$

a result which can be easily verified by seeking directly the law of force in a circle the centre of force being anywhere in its plane.

(2) Let the orbit be the reciprocal of a central conic with regard to its centre.

Here we have

$$P^2(a^2 + b^2 - R^2) = a^2b^2,$$

or $$\frac{K^4}{r^2}\left(a^2 + b^2 - \frac{K^4}{p^2}\right) = a^2b^2;$$

therefore $$\frac{K^8}{p^2} = (a^2 + b^2)K^4 - a^2b^2r^2;$$

therefore $$F dr = \tfrac{1}{2}h^2 d\left(\frac{1}{p^2}\right) = -\frac{h^2a^2b^2}{K^8} r dr,$$

and, therefore, $$F = -\frac{h^2 r}{\left(\frac{K^4}{a^2}\right)\left(\frac{K^4}{b^2}\right)}.$$

This result is *à priori* evident as the central reciprocal of a conic is a concentric conic whose semi-axes are the reciprocals of the semi-axes of the original one.

(iii) When the orbit is the pedal of a known curve with regard to the centre of force. With same notation as before we have $\frac{r}{p} = \frac{R}{P}$, with $r = P$; therefore, $\frac{r^2}{p} = R$, and, if

$R=f(P)$, we get $\frac{r^2}{p}=f(r)$, and hence, by the relation $Fdr=\frac{1}{2}h^2\,d\left(\frac{1}{p^2}\right)$, F may be obtained as a function of r; for example:

(1) Let the orbit be the pedal of a circle taken with regard to the centre of force which is situated at any point in its plane other than the centre.

If c be the distance of the centre of the circle from the centre of force and a the radius of the circle, then it is obvious, from geometrical considerations, that

$$c^2=a^2+R^2-2aP, \text{ or } R^2=c^2-a^2+2aP;$$

therefore
$$\frac{r^4}{p^2}=c^2-a^2+2ar;$$

therefore
$$\frac{1}{p^2}=\frac{c^2-a^2}{r^4}+\frac{2a}{r^3},$$

and, therefore,

$$Fdr=\tfrac{1}{2}h^2\,d\left(\frac{1}{p^2}\right)=h^2\left\{-\frac{3a}{r^4}+\frac{2\,(a^2-c^2)}{r^5}\right\}dr,$$

and
$$F=h^2\left\{-\frac{3a}{r^4}+\frac{2\,(a^2-c^2)}{r^5}\right\}.$$

When $c=a$ the centre of force is in the circumference of the circle, its pedal is the cardioid, and the law of force is $F=-\frac{3ah^2}{r^4}$.

(2) Let the orbit be the central pedal of a conic.

Here
$$a^2b^2=P^2\,(a^2+b^2-R^2),$$

therefore
$$a^2b^2=r^2\left(a^2+b^2-\frac{r^4}{p^2}\right),$$

or
$$\frac{1}{p^2}=(a^2+b^2)\,r^{-4}-a^2b^2r^{-6},$$

therefore
$$Fdr=\tfrac{1}{2}h^2d\left(\frac{1}{p^2}\right)=h^2\left\{\frac{3a^2b^2}{r^7}-\frac{2\,(a^2+b^2)}{r^5}\right\}dr,$$

and
$$F=h^2\left\{\frac{3a^2b^2}{r^7}-\frac{2\,(a^2+b^2)}{r^5}\right\},$$

the law here is the same as in (i) Ex. 2, which is explained by the fact that the inverse of a curve is the pedal of its

reciprocal, and, therefore, the central inverse of a conic is the central pedal of its reciprocal conic taken with regard to its centre.

Ex. 3. Let the orbit be the focal pedal of a conic.

Here $$P^2=\frac{b^2R}{2a-R}, \text{ or } r^2=\frac{\frac{b^2r^2}{p}}{2a-\frac{r^2}{p}};$$

therefore $$2a-\frac{r^2}{p}=\frac{b^2}{p},$$

or $$\frac{1}{p^2}=\frac{4a^2}{(b^2+r^2)^2};$$

therefore

$$Fdr=\tfrac{1}{2}h^2d\left(\frac{1}{p^2}\right)=\tfrac{1}{2}h^2d\left\{\frac{4a^2}{(b^2+r^2)^2}\right\}=8a^2h^2\left\{\frac{rdr}{(b^2+r^2)^3}\right\},$$

and $$F=\frac{8a^2h^2r}{(b^2+r^2)^3}.$$

As the focal pedal of a conic is the circle described on the axis major as diameter, the same result will be obtained if we seek the law of force in a circle of radius a described under the action of a force directed towards a fixed point at a distance $\sqrt{(a^2-b^2)}$ from the centre of the circle, and this may easily be verified; see II. ii. Ex. (1). The identity of the two results is explained by the remark at the foot of last page as the focal pedal of a conic is the inverse of its focal reciprocal, that is the inverse of a circle, viz. a circle.

3, Burlington-road, Dublin,
25th March, 1882.

MATHEMATICAL NOTES.

Théorèmes de Géométrie.

Appelons (o) un ellipsoide donné, c un point de cette surface et cn la normale en ce point.

Menons en c à l'ellipsoide une tangente quelconque ct et cherchons le point m où cette droite est touchée par une surface du second ordre (o') homofocale à (o).

La droite ct est une tangente commune à (o) et (o'). On sait que les plans qui touchent respectivement ces surfaces en c et en m sont rectangulaires. Le plan mené par ct et par la normale cn est alors le plan tangent en m à (o'). Pour déterminer m, nous sommes ainsi amenés à construire le point

où le plan *nct* est touché par une surface du second ordre homofocale à (*o*).

Appliquons ce théorème: les pôles d'un plan par rapport à des surfaces homofocales sont sur une droite perpendiculaire à ce plan. Le pied de cette perpendiculaire est alors le point *m* et comme nous connaissons sa direction nous n'avons qu'à en déterminer un point. Pour cela, il suffit de prendre, par rapport à (*o*), le pôle du plan normal *nct*.

Ainsi: *On prend le sommet du cône qui touche l'ellipsoide* (*o*) *suivant la courbe d'intersection de cette surface et du plan nct et l'on projette ce sommet sur la droite ct: le pied de cette perpendiculaire est le point m demandé.* Lorsque la droite *ct* tourne autour de *c*, en restant tangente à (*o*), il y a, pour chacune de ses positions un point tel que *m*: quel est le lieu de ces points? Pour le voir, il suffit de chercher combien il y a de points de ce lieu confondus en *c*. Le point *m* vient en *c*, lorsque *ct* est tangente à une ligne de courbure de (*o*) et comme il y a deux lignes de courbure qui passent en *c*, il y a en ce point deux points du lieu. Sur une tangente arbitraire telle que *ct*, il y a donc le point *m* et deux points confondus en *c*, par suite:

Le lieu des points tels que m est une courbe du 3e *ordre ayant un point double en c. Les tangentes à cette courbe en ce point sont rectangulaires.*

Projetons orthogonalement (*o*) sur son plan tangent en *c*, nous obtenons une courbe de contour apparent *A*. Le plan normal *nct* se projette suivant *ct*. Le sommet du cône circonscrit à (*o*) suivant la courbe d'intersection de cette surface et du plan *nct* n'est autre que le pôle de *ct* par rapport à *A*. Le théorème auquel nous venous d'arriver conduit alors à ce résultat de géométrie plane. *On donne sur un plan une conique A et un point fixe c. Sur les transversales qui passent par ce point, on projette respectivement les pôles de ces droites par rapport à A. Le lieu de ces projections est une courbe du* 3e *ordre qui a un point double en c et dont les tangentes en ce point sont rectangulaires.*

A. Mannheim.

On certain symbolic relations.

Since $$\frac{d}{dx}\, x^r.u = x^{r-1}\left(x\,\frac{d}{dx} + r\right) u,$$

we have, when $x = e^\theta$,

$$x^{1-r}\,\frac{d}{dx}\, x^r.u = \left(\frac{d}{d\theta} + r\right) u \quad \ldots\ldots\ldots\ldots (1).$$

Changing r with $r-1$ in the operator and applying it,

$$x^{2-r}\frac{d^2}{dx^2}x^r.u=\left(\frac{d}{d\theta}+r\right)\left(\frac{d}{d\theta}+r-1\right)u,$$

and, repeating the process,

$$x^{n-r}\frac{d^n}{dx^n}x^r.u=\left(\frac{d}{d\theta}+r\right)\left(\frac{d}{d\theta}+r-1\right)\ldots\left(\frac{d}{d\theta}+r-n+1\right)u\ldots(2).$$

By putting $r=0$, this equation gives the well-known result for $x^n\frac{d^n}{dx^n}$, and by putting $r=n$, it gives

$$\frac{d^n}{dx^n}x^n.u=\left(\frac{d}{d\theta}+n\right)\left(\frac{d}{d\theta}+n-1\right)\ldots\left(\frac{d}{d\theta}+1\right)u.$$

We see from (2) that symbols of the form $x^{n-r}\frac{d^n}{dx^n}x^r$ are commutative, and, for example, we have

$$\frac{d^m}{dx^m}x^{m+p}\frac{d^p}{dx^p}=x^p\frac{d^{m+p}}{dx^{m+p}}x^m,$$

and all symbols in which the sum of the exponents is equal to that of the indices of the derivatives are commutative.

As an example, we may take the formula*

$$\left(x^{-1}\frac{d}{dx}\right)^2x^5\left(x^{-1}\frac{d}{dx}\right)^3=\left(\frac{d}{dx}\right)^5,$$

which may be verified thus, the factors to be commuted being separated by points;

$$x^{-1}.\frac{d}{dx}x^{-1}\frac{d}{dx}x^3.x\frac{d}{dx}.x^{-1}\frac{d}{dx}x^{-1}\frac{d}{dx}$$

$$=\frac{d^2}{dx^2}x^{-1}.\frac{d}{dx}x.x\frac{d}{dx}.x^{-1}\frac{d}{dx}=\frac{d^5}{dx^5}.$$

As a second example, consider the formula

$$4\frac{d}{dx}x^3\frac{d}{dx}x.e^{x^2}=x^2\frac{d^4}{dx^4}e^{x^2}.$$

Since $\frac{d}{dx}e^{x^2}=2xe^{x^2}$, we have

$$4\frac{d}{dx}x^3\frac{d}{dx}xe^{x^2}=2\frac{d}{dx}x^3\frac{d^2}{dx^2}e^{x^2}=2x^2\frac{d^3}{dx^3}x.e^{x^2}=x^2\frac{d^4}{dx^4}e^{x^2}.$$

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* *Proceedings of the London Mathematical Society*, vol. VIII. p. 51.

END OF VOL. XI.

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